Relational Logic Semantics

Propositional Logic Semantics

A Propositional logic *interpretation* is an association between the propositional constants in a propositional language and the truth values T or F.
Relational Logic Semantics

The *big question*: what is a relational logic interpretation? There are no proposition constants, just object constants, function constants, and relation constants. To what do they refer?

Universe of Discourse

A *Universe of Discourse* is a set of objects about which we want to say something.

Examples:

- \{1, 2, 3, 4, \ldots\}
- \{0, 1, -1, 2, -2, \ldots\}
- Set of real numbers
- Set of complex numbers
- \{washington, jefferson, \ldots, bush, obama\}
- \{□, ●, ★, ●, ♦\}
Relations

Given a universe of discourse $U$, a relation is a set of $n$-tuples of objects in $U$ each of which manifests a particular property or relationship.

Sample Universe of Discourse: \{1, 2, 3, 4\}

Example of a 2-ary (binary) relation:
\{(\langle 1,2 \rangle), (\langle 1,3 \rangle), (\langle 1,4 \rangle), (\langle 2,3 \rangle), (\langle 2,4 \rangle), (\langle 3,4 \rangle)\}

Arity

Each relation has an arity that determines the number of objects that can participate in an instance of the relation.

Arity 1 - unary relation
\{(\langle 1 \rangle), (\langle 3 \rangle)\} or, more simply, \{1, 3\}

Arity 2 - binary relation
\{(\langle 1,2 \rangle), (\langle 1,3 \rangle), (\langle 1,4 \rangle), (\langle 2,3 \rangle), (\langle 2,4 \rangle), (\langle 3,4 \rangle)\}

Arity 3 - ternary relation
\{(\langle 1,2,3 \rangle), (\langle 1,3,4 \rangle), (\langle 2,2,4 \rangle)\}
Cardinality

The cardinality of a relation is the number of tuples in the relation.

Cardinality 2

\{1, 3\}

Cardinality 6

\{\langle 1,2\rangle, \langle 1,3\rangle, \langle 1,4\rangle, \langle 2,3\rangle, \langle 2,4\rangle, \langle 3,4\rangle\}

Cardinality 3

\{\langle 1,2,3\rangle, \langle 1,3,4\rangle, \langle 2,2,4\rangle\}

Counting

Assume a universe of discourse with 4 objects.
Number of 2-tuples: \(4^2=16\)
Number of binary relations: \(2^{16}\)

Assume a universe of discourse with \(n\) objects.
Number of \(k\)-tuples: \(n^k\)
Number of \(k\)-ary relations: \(2^{n^k}\)

Question: How many 0-ary relations are there?
Functions

Given a universe of discourse, an $n$-ary function is a relation associating each combination of $n$ objects in a universe of discourse (called the arguments) with a single object (called the value).

Universe of Discourse: \{1, 2, 3, 4\}

Example:

\[
\begin{align*}
1 & \mapsto 2 \\
2 & \mapsto 3 \\
3 & \mapsto 4 \\
4 & \mapsto 1 \\
\end{align*}
\]

Total and Single-Valued

Functions are total and single-valued - one and exactly one value for each combination of arguments.

Partial - not defined for some combination of arguments

Multivalued - more than value for some argument combination

NB: We ignore partial and multi-valued functions.
Unary Functions as Binary Relations

Function as association:
1 → 2
2 → 3
3 → 4
4 → 1

Function as relation:
\{<1,2>, <2,3>, <3,4>, <4,1>\}

Alternative Notation:
\{<1 \mapsto 2>, <2 \mapsto 3>, <3 \mapsto 4>, <4 \mapsto 1>\}

Counting

Assume a universe of discourse with 4 objects.
Number of 1-tuples: 4
Number of unary functions: \(4^4 = 256\)
Number of binary relations: \(2^{16} = 65536\)

Assume a universe of discourse with \(n\) objects.
Number of \(k\)-tuples: \(n^k\)
Number of \(k\)-ary functions: \(n^k = 2^{n^k \log n}\)
Number of \(k+1\)-ary relations: \(2^{n^{k+1}} = 2^{n^k n}\)

Quiz: How many 0-ary functions are there?
Role of Logic

Incomplete Information
Block $a$ is on block $b$ or it is on block $c$.
Block $a$ is not on block $b$.

Integrity
A block may not be $on$ itself.
A block may be $on$ only one block at a time.

Definitions
A block is $under$ another iff the second is $on$ the first.
A block is $clear$ iff there is no block $on$ it.
A block is on the $table$ iff there is no block $under$ it.

Interpretations
An interpretation is a mapping $i$ that assigns “meaning” to the elements of a signature
$\langle a_1, \ldots, a_k, f_1, \ldots, f_m, r_1, \ldots, r_n \rangle$ in terms of a universe of discourse $U$.

Object Constants:
$i(a_j) \in U$

Function Constants:
$i(f_j^n): U^m \rightarrow U$

Relation Constants:
$i(r_j^n) \subseteq U^m$
Example

\[ |i| = U = \{1, 2, 3\} \]

\[ i(a) = 1 \]
\[ i(b) = 2 \]
\[ i(c) = 2 \]

\[ i(f) = \{1 \rightarrow 2, 2 \rightarrow 3, 3 \rightarrow 3\} \]

\[ i(p) = \{\langle 1 \rangle\} \]
\[ i(q) = \{\langle 1, 2 \rangle, \langle 1, 1 \rangle, \langle 2, 2 \rangle\} \]
\[ i(r) = \{\langle 1, 2, 1 \rangle, \langle 2, 2, 1 \rangle\} \]

Ground Value Assignments

A ground value assignment \( s_i \) based on interpretation \( i \) is a mapping from the terms of the language into the universe of discourse. \( s_i \) must agree with \( i \) on constants; and, for functional terms, it yields the result of applying the interpretation of the function constant to the values assigned to the argument terms.

\[ s_i(\sigma) = i(\sigma) \]
\[ s_i(\pi(\tau_1, \ldots, \tau_n)) = i(\pi(s_i(\tau_1), \ldots, s_i(\tau_n))) \]
Example

Interpretation:
\[ i(a) = 1 \]
\[ i(b) = 2 \]
\[ i(f) = \{1 \rightarrow 2, 2 \rightarrow 1\} \]
\[ i(r) = \{\langle 1,2 \rangle, \langle 2,2 \rangle\} \]

Value Assignment:
\[ s_i(a) = i(a) = 1 \]
\[ s_i(f(a)) = i(f)(s_i(a)) = i(f)(1) = 2 \]

Ground Truth Assignments

A ground truth assignment \( t_i \) based on interpretation \( i \) is a mapping from the sentences of the language into \{true, false\}.

\[ t_i: \text{sentence} \rightarrow \{\text{true}, \text{false}\} \]

The details of the definition are given on the following slides.
Relational Sentences

A ground truth assignment satisfies a relational sentence if and only if the tuple of objects denoted by the arguments is a member of the relation denoted by the relation constant.

\[
t_i(\rho(\tau_1, \ldots, \tau_n)) = \begin{cases} 
true & \text{if } \langle s_{i_1}(\tau_1), \ldots, s_{i_n}(\tau_n) \rangle \in i(\rho) \\
false & \text{otherwise}
\end{cases}
\]

Example

Interpretation:
\[
i(a) = 1 \\
i(b) = 2 \\
i(f) = \{1\rightarrow2, 2\rightarrow1\} \\
i(r) = \{\langle1,2\rangle, \langle2,2\rangle\}
\]

Example:
\[
t_i(r(a,b)) = true \text{ since } \langle1,2\rangle \in i(r) \\
t_i(r(b,a)) = false \text{ since } \langle2,1\rangle \notin i(r)
\]
Logical Sentences

\[ t_{iv}(\neg \varphi) = true \text{ iff } t_{iv}(\varphi) = false \]

\[ t_{iv}(\varphi \land \psi) = true \text{ iff } t_{iv}(\varphi) = true \text{ and } t_{iv}(\psi) = true \]

\[ t_{iv}(\varphi \lor \psi) = true \text{ iff } t_{iv}(\varphi) = true \text{ or } t_{iv}(\psi) = true \]

\[ t_{iv}(\varphi \Rightarrow \psi) = true \text{ iff } t_{iv}(\varphi) = false \text{ or } t_{iv}(\psi) = true \]

\[ t_{iv}(\varphi \Leftarrow \psi) = true \text{ iff } t_{iv}(\varphi) = true \text{ or } t_{iv}(\psi) = false \]

\[ t_{iv}(\varphi \Leftrightarrow \psi) = true \text{ iff } t_{iv}(\varphi) = t_{iv}(\psi) \]

Variable Assignments

A *variable assignment* for a universe of discourse \( U \) is a function assigning variables to objects in \( U \).

\[ v: \text{Variable} \rightarrow U \]

Universe of Discourse:

\( U = \{1, 2, 3\} \)

Example:

\[ v(x) = 1 \quad v(x) = 2 \]
\[ v(y) = 2 \quad v(y) = 2 \]
\[ v(z) = 3 \quad v(z) = 2 \]
Value Assignments

A value assignment $s_{iv}$ based on interpretation $i$ and variable assignment $v$ is a mapping from the terms of the language into the universe of discourse. $s_{iv}$ must agree with $i$ on constants it must agree with $v$ on variables; and, for functional terms, it yields the result of applying the interpretation of the function constant to the values assigned to the argument terms.

$$s_{iv}(\sigma) = i(\sigma)$$
$$s_{iv}(\upsilon) = w(\upsilon)$$
$$s_{iv}(\pi(\tau_1, \ldots, \tau_n)) = i(\pi)(s_{iv}(\tau_1), \ldots, s_{iv}(\tau_n))$$

Truth Assignments

A truth assignment $t_{iv}$ based on interpretation $i$ and variable assignment $v$ is a mapping from the sentences of the language into \{true, false\}.

$$t_{iv}: \text{sentence} \rightarrow \{\text{true}, \text{false}\}$$

The details of the definition are given on the following slides.
Relational Sentences

A truth assignment satisfies a relational sentence if and only if the tuple of objects denoted by the arguments is a member of the relation denoted by the relation constant.

\[ t_i(\rho(\tau_1, \ldots, \tau_n)) = true \text{ if and only if } \langle s_i(\tau_1), \ldots, s_i(\tau_1) \rangle \in i(\rho) \]
\[ = false \text{ otherwise} \]

Logical Sentences

\[ t_i(\neg \phi) = true \text{ iff } t_i(\phi) = false \]
\[ t_i(\phi \land \psi) = true \text{ iff } t_i(\phi) = true \text{ and } t_i(\psi) = true \]
\[ t_i(\phi \lor \psi) = true \text{ iff } t_i(\phi) = true \text{ or } t_i(\psi) = true \]
\[ t_i(\phi \Rightarrow \psi) = true \text{ iff } t_i(\phi) = false \text{ or } t_i(\psi) = true \]
\[ t_i(\phi \Leftrightarrow \psi) = true \text{ iff } t_i(\phi) = true \text{ or } t_i(\psi) = false \]
\[ t_i(\phi \Leftrightarrow \psi) = true \text{ iff } t_i(\phi) = t_i(\psi) \]
Quantified Sentences

Intuitively, a universally quantified sentence is true if and only if it is true no matter what value we assign to the universally quantified variable.

Intuitively, an existentially quantified sentence is true if and only if it is true for some value of the existentially quantified variable.

Stating these definitions precisely is a little tricky due to the possibility of nested quantifiers.

$$\forall x. (\exists y. r(x,y) \Rightarrow \forall x. r(x,x))$$

Versions

A version $v[\omega \leftarrow x]$ of a variable assignment $v$ is the variable assignment that agrees with $v$ on all variables except $\omega$, which is assigned the value $x$.

$$v[\omega \leftarrow x](\theta) = x \quad \text{if } \theta = \omega$$
$$v[\omega \leftarrow x](\theta) = v(\theta) \quad \text{if } \theta \neq \omega$$
Examples

<table>
<thead>
<tr>
<th>Interpretation:</th>
<th>Variable Assignment:</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td>i</td>
</tr>
<tr>
<td></td>
<td>$v(y) = 2$</td>
</tr>
<tr>
<td>Version 1:</td>
<td></td>
</tr>
<tr>
<td>$v<a href="x">x \leftarrow 1</a> = 1$</td>
<td></td>
</tr>
<tr>
<td>$v<a href="y">x \leftarrow 1</a> = 2$</td>
<td></td>
</tr>
<tr>
<td>Version 2:</td>
<td></td>
</tr>
<tr>
<td>$v<a href="x">x \leftarrow 2</a> = 2$</td>
<td></td>
</tr>
<tr>
<td>$v<a href="y">x \leftarrow 2</a> = 2$</td>
<td></td>
</tr>
<tr>
<td>Version 3:</td>
<td></td>
</tr>
<tr>
<td>$v<a href="x">y \leftarrow 1</a> = 1$</td>
<td></td>
</tr>
<tr>
<td>$v<a href="y">y \leftarrow 1</a> = 1$</td>
<td></td>
</tr>
<tr>
<td>Version 4:</td>
<td></td>
</tr>
<tr>
<td>$v<a href="x">y \leftarrow 2</a> = 1$</td>
<td></td>
</tr>
<tr>
<td>$v<a href="y">y \leftarrow 2</a> = 2$</td>
<td></td>
</tr>
</tbody>
</table>

Quantified Sentences

A universally quantified sentence is true in interpretation $i$ and variable assignment $v$ if and only if the scope is true for $i$ and every version of $v$.

$$t_i(\forall \omega. \varphi) = \text{true} \text{ iff } t_{iv[\omega \leftarrow x]}(\varphi) = \text{true} \text{ for all } x \in |i|.$$  

An existentially quantified sentence is true in interpretation $i$ and variable assignment $v$ if and only if the scope is true for $i$ and some version of $v$.

$$t_i(\exists \omega. \varphi) = \text{true} \text{ iff } t_{iv[\omega \leftarrow x]}(\varphi) = \text{true} \text{ for some } x \in |i|.$$
Examples

Interpretation:

<table>
<thead>
<tr>
<th>i</th>
<th>=</th>
<th>{1,2}</th>
</tr>
</thead>
<tbody>
<tr>
<td>i(a)</td>
<td>=</td>
<td>1</td>
</tr>
<tr>
<td>i(b)</td>
<td>=</td>
<td>2</td>
</tr>
<tr>
<td>i(r)</td>
<td>=</td>
<td>{〈1, 2〉, 〈2, 2〉}</td>
</tr>
</tbody>
</table>

| t iv (∀x.r(x,x)) = ? |
| t iv[x←1](r(x,x)) = true |
| t iv[x←2](r(x,x)) = true |

Variable Assignment:

| iil | = | {1,2} |
| v(x) | = | 1 |
| v(y) | = | 2 |

| t iv (∃x.r(x,x)) = ? |
| t iv[x←1](r(x,x)) = false |
| t iv[x←2](r(x,x)) = true |

Examples

Interpretation:

<table>
<thead>
<tr>
<th>i</th>
<th>=</th>
<th>{1,2}</th>
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<td>i(r)</td>
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<td>{〈1, 2〉, 〈2, 2〉}</td>
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</table>

| t iv (∀x.∃y.r(x,y)) = ? |
| t iv[x←1](∃y.r(x,y)) = true |
| t iv[x←2](∃y.r(x,y)) = true |

Variable Assignment:

| iil | = | {1,2} |
| v(x) | = | 1 |
| v(y) | = | 2 |

| t iv (∃x.∀y.r(x,y)) = ? |
| t iv[x←1](∀y.r(x,y)) = false |
| t iv[x←2](∀y.r(x,y)) = false |
Models

A sentence is true in interpretation $i$ and variable assignment $v$ if and only if it satisfies the rules we have just seen.

An arbitrary sentence is true in interpretation $i$ if and only if it is true for $i$ and every compatible variable assignment $v$.

An interpretation of a sentence is a model if and only if the sentence is true in that interpretation.

Open and Closed Sentences

An occurrence of a variable is free if and only if it does not lie in the scope of any quantifier. A closed sentence is a sentence with no free occurrences of variables. Otherwise, the sentence is open.
Observation

Observation: If an interpretation satisfies a closed sentence for one variable assignment, it satisfies the sentence for every variable assignment (a model).

\[ \models_i \forall x. \exists y. r(x, y) \; [v_1] \]

iff \[ \models_i \forall x. \exists y. r(x, y) \; [v_2] \]

... iff \[ \models_i \forall x. \exists y. r(x, y) \; [v_n] \]

Observation

An interpretation is a model of an open sentence iff it is a model of the sentence obtained by universally quantifying all of the free variables.

\[ \models_i r(x, y) \text{ iff } \models_i \forall x. \forall y. r(x, y) \]
Properties of Sentences

<table>
<thead>
<tr>
<th></th>
<th>A sentence is valid if and only if every interpretation satisfies it.</th>
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</thead>
<tbody>
<tr>
<td>Valid</td>
<td></td>
</tr>
<tr>
<td></td>
<td>A sentence is contingent if and only if some interpretation satisfies it and some interpretation falsifies it.</td>
</tr>
<tr>
<td>Contingent</td>
<td></td>
</tr>
<tr>
<td></td>
<td>A sentence is unsatisfiable if and only if no interpretation satisfies it.</td>
</tr>
<tr>
<td>Unsatisfiable</td>
<td></td>
</tr>
</tbody>
</table>

Properties of Sentences

<table>
<thead>
<tr>
<th></th>
<th>A sentence is satisfiable if and only if it is either valid or contingent.</th>
</tr>
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<tbody>
<tr>
<td>Valid</td>
<td></td>
</tr>
<tr>
<td></td>
<td>A sentence is falsifiable if and only if it is contingent or unsatisfiable.</td>
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<td>Contingent</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>Unsatisfiable</td>
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</tbody>
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Structures

We can view an interpretation as vector consisting of a universe of discourse and values for the items in the signature of the language (when the signature is ordered).

In what follows, we call such vectors structures.

Example

Interpretation:
\begin{align*}
l & = \{1, 2\} \\
i(a) & = 1 \\
i(b) & = 2 \\
i(f) & = \{1 \rightarrow 2, 2 \rightarrow 1\} \\
i(r) & = \{\langle 1,2 \rangle, \langle 1,1 \rangle, \langle 2,2 \rangle\}
\end{align*}

Signature:
\[ \langle a, b, f, r \rangle \]

Structure:
\[ \langle \{1, 2\}, 1, 2, \{1 \rightarrow 2, 2 \rightarrow 1\}, \{\langle 1,2 \rangle, \langle 1,1 \rangle, \langle 2,2 \rangle\}\rangle \]
Elementary Equivalence

Two structures are *elementarily equivalent* if and only if they satisfy the same set of sentences for all signatures.

NB: There are pairs of structures that cannot be distinguished from each other in Relational Logic.

Examples

\[\langle \{1,2\}, 1, 2, \{\langle 1,2 \rangle, \langle 2,1 \rangle\}\rangle\]
\[\langle \{3,4\}, 3, 4, \{\langle 3,4 \rangle, \langle 4,3 \rangle\}\rangle\]
\[\langle \{1,2\}, 1, 2, \{\langle 1,2 \rangle, \langle 2,1 \rangle\}\rangle\]
\[\langle \{1,2\}, 2, 1, \{\langle 1,2 \rangle, \langle 2,1 \rangle\}\rangle\]

\[\langle Q, < \rangle\]
\[\langle R, < \rangle\]
Lowenheim Skolem Tarski Theorem

If there is a model of a set of first-order relational sentences of any infinite cardinality, then there is a model of every infinite cardinality.

Transitivity Theorem

It is not possible in first-order logic to define transitive closure in first-order logic.

More precisely, it is not possible characterize the set of structures consisting of an arbitrary universe, an arbitrary binary relation, and the transitive closure of that relation.