Two sources of explosion

Eric Jui-Yi Kao
with Carl Hewitt

Department of Computer Science
Stanford University
Stanford, CA 94305 United States

erickao@cs.stanford.edu

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Inconsistency Robustness 2011

Inconsistency Robustness

- Automated logical reasoning form a part of many systems.
  - security policy systems
  - semantic web
  - knowledge bases
- Some logics are explosive

  I.E. $\{\alpha, \neg\alpha\} \vdash \beta$, for any sentences α, β.
- Non-explosion is a minimal requirement for inconsistency robustness.

Classical logic is explosive

1	α	Premise
2	$\neg\alpha$	Premise
3	$\quad \neg\beta$	
4	$\quad \alpha$	Reiteration, 1
5	$\quad \neg\alpha$	Reiteration, 2
6	$\quad \bot$	Contradiction, 4, 5
7	$\neg\neg\beta$	Proof by contradiction, 3–6
8	β	Double negation elimintation, 7

Classical logic is explosive

1	α	Premise
2	$\neg\alpha$	Premise
3	$\beta \vee \alpha$	$\vee$-Introduction, 1
4	β	Disjunctive syllogism, 1, 3

Outline

- Idea: restrict the proof theory of classical logic in some "reasonable" way
  - Avoid explosion
  - Retain "maximal" deductive power.
- Many "design decisions" involved
  e.g., cannot retain both $\vee$-introduction and disjunctive syllogism
- Direct Logic is one proposal [2]
- Can we increase its deductive power?
- We consider two attempts in increasing its deductive power

Boolean Direct Logic rules of inference

Core rules of bDL

$$\frac{\alpha \vee \beta \qquad \neg\alpha \vee \psi}{\beta \vee \psi}\text{[Resolution]}$$

$$\frac{\alpha \wedge \beta}{\alpha}\text{[}\wedge\text{-Elimination]}$$

$$\frac{\alpha \qquad \beta}{\alpha \vee \beta}\text{[Restricted }\vee\text{-Introduction]}$$

$$\frac{\alpha \qquad \beta}{\alpha \wedge \beta}\text{[}\wedge\text{-Introduction]}$$

Substitution according boolean equivalences

$$\frac{\alpha}{s(\alpha)} \text{ [Substitution]},$$

where $s(\alpha)$ is the result of substituting in α an occurrence of a subformula by an equivalent subformula according to a boolean equivalence below.

Distributivity	$\psi \vee (\alpha \wedge \beta)$	$\equiv$	$(\psi \vee \alpha) \wedge (\psi \vee \beta)$
	$(\psi \wedge \alpha) \vee (\psi \wedge \beta)$	$\equiv$	$\psi \wedge (\alpha \vee \beta)$
De Morgan Laws	$\neg(\alpha \wedge \beta)$	$\equiv$	$\neg\alpha \vee \neg\beta$
	$\neg(\alpha \vee \beta)$	$\equiv$	$\neg\alpha \wedge \neg\beta$
Double negation	$\neg\neg\alpha$	$\equiv$	α
Idempotence	$\alpha \vee \alpha$	$\equiv$	α
	$\alpha \wedge \alpha$	$\equiv$	α

Some properties of bDL

- bDL is not explosive
- bDL is "reasonable"
- Can we make bDL more powerful?
  e.g., bDL cannot prove $p \vee \neg p$

Law of excluded middle

- Intuitively: no sentence can be neither true nor false.
- Axiom schema

$$\alpha \vee \neg\alpha \text{ [Excluded Middle]}$$

- Not obvious whether bDL+[Excluded Middle] is explosive

E.G. bDL plus the axioms $\{p \vee \neg p : p \in \textsc{Propositions}\}$ is not explosive [5].

Excluded Middle is explosive

1	α	Premise
2	$\neg\alpha$	Premise
3	$(\alpha \wedge \neg\beta) \vee \neg(\alpha \wedge \neg\beta)$	Excluded Middle
4	$(\alpha \wedge \neg\beta) \vee \neg\alpha \vee \neg\neg\beta$	De Morgan, 3
5	$(\alpha \wedge \neg\beta) \vee \neg\alpha \vee \beta$	Double negation, 4
6	$(\alpha \vee \neg\alpha \vee \beta) \wedge (\neg\beta \vee \neg\alpha \vee \beta)$	Distributivity, 5
7	$\alpha \vee \neg\alpha \vee \beta$	$\wedge$-Elimination, 6
8	$\alpha \vee \beta$	Resolution, 7, 1
9	β	Resolution, 8, 2

Proof by self-refutation

- Intuitively: If a sentence negates itself, it must be false.
- If a sentence α derives the negation of itself, then we can introduce $\neg\alpha$.
- Axiom schema:

$$\neg\alpha, \text{ where } \alpha \text{ proves } \neg\alpha \text{ [Self-Refutation]}$$

Proof 2a: $((\neg\alpha \vee \neg\beta) \wedge (\alpha \vee \beta))$ proves $\neg((\neg\alpha \vee \neg\beta) \wedge (\alpha \vee \beta))$

1	$(\neg\alpha \wedge \neg\beta) \wedge (\alpha \vee \beta)$	Premise
2	$(\neg\alpha \wedge \neg\beta)$	$\wedge$-Elimination, 1
3	$(\alpha \vee \beta)$	$\wedge$-Elimination, 1
4	$(\alpha \vee \beta) \vee (\neg\alpha \wedge \neg\beta)$	Restricted $\vee$-Introduction, 2, 3
5	$(\alpha \vee \beta) \vee \neg(\alpha \vee \beta)$	De Morgan, 4
6	$(\alpha \vee \neg\neg\beta) \vee \neg(\alpha \vee \beta)$	Double negation, 5
7	$(\neg\neg\alpha \vee \neg\neg\beta) \vee \neg(\alpha \vee \beta)$	Double negation, 6
8	$\neg(\neg\alpha \vee \neg\beta) \vee \neg(\alpha \vee \beta)$	De Morgan, 7
9	$\neg((\neg\alpha \vee \neg\beta) \wedge (\alpha \vee \beta))$	De Morgan, 8

1	α	Premise
2	$\neg\alpha$	Premise
3	$\neg((\neg\alpha \vee \neg\beta) \wedge (\alpha \vee \beta))$	Self-Refutation, Proof 2a
4	$\neg(\neg\alpha \wedge \neg\beta) \vee \neg(\alpha \vee \beta)$	De Morgan, 3
5	$(\neg\neg\alpha \vee \neg\neg\beta) \vee \neg(\alpha \vee \beta)$	De Morgan, 4
6	$(\neg\neg\alpha \vee \beta) \vee \neg(\alpha \vee \beta)$	Double negation, 5
7	$\alpha \vee \beta \vee \neg(\alpha \vee \beta)$	Double negation, 6
8	$\alpha \vee \beta \vee (\neg\alpha \wedge \neg\beta)$	De Morgan, 7
9	$(\alpha \vee \beta \vee \neg\alpha) \wedge (\alpha \vee \beta \vee \neg\beta)$	Distributivity, 8
10	$\alpha \vee \beta \vee \neg\alpha$	$\wedge$-Elimination, 9
11	$\alpha \vee \beta$	Resolution, 10, 1
12	β	Resolution, 11, 2

Design decisions

- Let's take the boolean equivalences and $\wedge$-Elimination for granted
- The explosiveness of bDL+[Excluded Middle] essentially rely on only
  - Excluded Middle and
  - Disjunctive Syllogism (a special case of Resolution)
- Direct Logic chooses Disjunctive Syllogism and leaves out Excluded Middle
- The explosiveness of bDL+[Self-Refutation] essentially rely on only
  - Self-Refutation,
  - Disjunctive Syllogism, and
  - Restricted $\vee$-Introduction ($\alpha \vee \beta$ from α and β)
- Direct Logic replaces Self-Refutation with a weaker rule.

Other logics

- The results apply to other paraconsistent logics that support the rules used.
- For example, Besnard and Hunter's quasi-classical logic [1, 4, 3] also becomes explosive if either Excluded Middle or Self-Refutation is added.

Open questions

Consider the set of valid inference rules in classical boolean logic:

$$\mathcal{R} = \{\frac{\Phi_1 \cdots \Phi_n}{\Psi} : \phi_1 \cdots \phi_n \models \psi$$

for any intances $\phi_1, \ldots, \phi_n, \psi$ of $\Phi_1, \ldots, \Phi_n, \Psi\}$

- Find a maximal subset S of $\mathcal{R}$ such that the the logic induced by S is not explosive.
- Can the induced logic be axiomatized by a finite number of inference rules?
- Is the induced logic decidable?
- Characterize the space of all such $S \subseteq \mathcal{R}$

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Proof by contradiction

- If by assuming a sentence α we derive a contradiction, then we can conclude $\neg\alpha$.
- It can be stated as the following meta-rule:
  If $\Sigma, \alpha \vdash \psi$ and $\Sigma, \alpha \vdash \neg\psi$, then conclude $\Sigma \vdash \neg\alpha$.
- Proof by contradiction easily leads to explosiveness. For any sentences α and β,

$$\{\alpha, \neg\alpha\}, \neg\beta \vdash \alpha \text{ and } \{\alpha, \neg\alpha\}, \neg\beta \vdash \neg\alpha,$$

  hence $\{\alpha, \neg\alpha\} \vdash \neg\neg\beta$ using proof by contradiction.

Self-refutation is explosive

I show that the addition of the proof by self-refutation rule to bDL leads to explosiveness.
For any pair of sentences α and β, I derive β from premises α and $\neg\alpha$, using bDL inference rules plus the Self-Refutation axiom schema.
First, I show that $(\neg\alpha \wedge \neg\beta) \wedge (\alpha \vee \beta)$ proves its own negation $\neg((\neg\alpha \vee \neg\beta) \wedge (\alpha \vee \beta))$. Then I use $\neg((\neg\alpha \vee \neg\beta) \wedge (\alpha \vee \beta))$, α, and $\neg\alpha$ to prove β.