Two sources of explosion

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Abstract. In pursuit of enhancing the deductive power of Direct Logic while avoiding explosiveness, Hewitt has proposed including the law of excluded middle and proof by self-refutation. In this paper, I show that the inclusion of either one of these inference patterns causes paraconsistent logics such as Hewitt's Direct Logic and Besnard and Hunter's quasi-classical logic to become explosive.

1 Introduction

A central goal of a paraconsistent logic is to avoid explosiveness – the inference of any arbitrary sentence β from an inconsistent premise set $\{p, \neg p\}$ (*ex falso quodlibet*).

Hewitt [2] Direct Logic and Besnard and Hunter's quasi-classical logic (QC) [1, 5, 4] both seek to preserve the deductive power of classical logic "as much as possible" while still avoiding explosiveness. Their work fits into the ongoing research program of identifying some "reasonable" and "maximal" subsets of classically valid rules and axioms that do not lead to explosiveness.

To this end, it is natural to consider which classically sound deductive rules and axioms one can introduce into a paraconsistent logic without causing explosiveness. Hewitt [3] proposed including the law of excluded middle and the proof by self-refutation rule (a very special case of proof by contradiction) but did not show whether the resulting logic would be explosive.

In this paper, I show that for quasi-classical logic and its variant, the addition of either the law of excluded middle or the proof by self-refutation rule in fact leads to explosiveness.

I first introduce bDL [2], boolean fragment of Direct Logic (section 2). In section 3, I discuss the law of excluded middle and show that its addition to bDL leads to explosiveness. In section 4, I discuss proof by self-refutation and show that its addition to bDL leads to explosiveness. In section 5, I discuss how the results also hold in quasi-classical logic (QC).

2 Boolean Direct Logic

Boolean Direct Logic (bDL) is a boolean fragment of Direct Logic [2]. It allows us to analyze some of the essential structure of Direct Logic without engaging the complexity of the full logic.

2.1 Formal definition of bDL

The language of bDL is the language of boolean logic.

Definition 1. *Let the language $\mathcal{L}$ be the set of classical boolean propositional formulas (of finite length) formed from a set of propositional atoms $\mathcal{A}$ and the connectives $\wedge, \vee, \neg$. $\mathcal{L}$ is the fixed language assumed throughout this paper.*

The proof theory of bDL is presented as a set rules of inference. There are four core rules, supplemented by the rules of substitution using the usual boolean equivalences.

Definition 2 (bDL rules of inference). *The following are the bDL rules of inference. Conjunction and disjunction are taken to be commutative and associative.*

Core rules of bDL

$$\frac{\alpha \vee \beta \qquad \neg\alpha \vee \psi}{\beta \vee \psi}\text{[Resolution]} \qquad \frac{\alpha \wedge \beta}{\alpha}\text{[}\wedge\text{-Elimination]}$$

$$\frac{\alpha \qquad \beta}{\alpha \vee \beta}\text{[Restricted }\vee\text{-Introduction]} \qquad \frac{\alpha \qquad \beta}{\alpha \wedge \beta}\text{[}\wedge\text{-Introduction]}$$

Substitution according boolean equivalences

$\frac{\alpha}{s(\alpha)}$, where $s(\alpha)$ is the result of substituting in α an occurrence of a subformula by an equivalent subformula according to a boolean equivalence below.

Distributivity	$\psi \vee (\alpha \wedge \beta)$	is equivalent to	$(\psi \vee \alpha) \wedge (\psi \vee \beta)$
	$(\psi \wedge \alpha) \vee (\psi \wedge \beta)$	is equivalent to	$\psi \wedge (\alpha \vee \beta)$
De Morgan Laws	$\neg(\alpha \wedge \beta)$	is equivalent to	$\neg\alpha \vee \neg\beta$
	$\neg(\alpha \vee \beta)$	is equivalent to	$\neg\alpha \wedge \neg\beta$
Double negation	$\neg\neg\alpha$	is equivalent to	α
Idempotence	$\alpha \vee \alpha$	is equivalent to	α
	$\alpha \wedge \alpha$	is equivalent to	α

Definition 3 (bDL proof). *A bDL proof is a sequence of sentences $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that each α_i is a premise or follows from $\alpha_1, \ldots, \alpha_{i-1}$ by the application of a bDL rule of inference.*

Definition 4 (bDL consequence). *A sentence β is a bDL-consequence of a set of sentences Σ if there exists a bDL proof $\alpha_1, \alpha_2, \ldots, \alpha_k, \beta$ with premises Σ. This fact is denoted by $\Sigma \vdash_{\text{bDL}} \beta$.*

3 Law of excluded middle

Intuitively, the law of excluded middle states that no sentence can be neither true nor false.

Some authors have suggested incorporating the law of excluded middle into a paraconsistent logic to mimic classical reasoning [3].

Restall [7] clarified the connection between the law of excluded middle and explosiveness in pointing out that explosiveness is one particular "implementation" of the excluded middle principle. But explosiveness is not a necessary consequence of the law of excluded middle. In fact, there are paraconsistent logics that support the law of excluded middle without being explosive [7].

The law can be formally introduced into a logic by providing the axiom schema

$$\alpha \lor \neg\alpha \text{ [Excluded Middle]}$$ [1]

that sentences of this form may be introduced without justification into a proof.

It is not obvious whether introducing the Excluded Middle axiom schema into bDL would lead to explosiveness. For example, bDL plus the axioms $\{p \lor \neg p : p \in \mathcal{A}\}$ is not explosive [6].

I show that the law of excluded middle in fact leads to explosiveness in bDL. Specifically, for any sentences α and β, I derive β from premises α and $\neg\alpha$ using the rules of bDL plus the Excluded Middle axiom schema.

Proof 1: Explosion proof in bDL+[Excluded Middle]

1	α	Premise
2	$\neg\alpha$	Premise
3	$(\alpha \land \neg\beta) \lor \neg(\alpha \land \neg\beta)$	Excluded Middle
4	$(\alpha \land \neg\beta) \lor \neg\alpha \lor \neg\neg\beta$	De Morgan, 3
5	$(\alpha \land \neg\beta) \lor \neg\alpha \lor \beta$	Double negation, 4
6	$(\alpha \lor \neg\alpha \lor \beta) \land (\neg\beta \lor \neg\alpha \lor \beta)$	Distributivity, 5
7	$\alpha \lor \neg\alpha \lor \beta$	$\land$-Elimination, 6
8	$\alpha \lor \beta$	Resolution, 7, 1
9	β	Resolution, 8, 2

4 Proof by self-refutation

In this section, I introduce the proof by self-refutation rule as a special case of proof by contradiction.

[1] In logics where $\alpha \rightarrow \beta$ is equivalent to $\neg\alpha \lor \beta$, the [Excluded Middle] axiom schema is equivalent to $\alpha \rightarrow \alpha$.

The proof by contradiction rule[2] states that if by assuming a sentence α we derive a contradiction, then we can conclude $\neg\alpha$. It can be stated as the following meta rule: If $\Sigma, \alpha \vdash \psi$ and $\Sigma, \alpha \vdash \neg\psi$, then conclude $\Sigma \vdash \alpha$.

Proof by contradiction easily leads to explosiveness. For any sentences α and β,

$$\{\alpha, \neg\alpha\}, \neg\beta \vdash \alpha \text{ and } \{\alpha, \neg\alpha\}, \neg\beta \vdash \neg\alpha,$$

hence $\{\alpha, \neg\alpha\} \vdash \neg\neg\beta$ using proof by contradiction.

The proof by self-refutation rule states that if a sentence α derives the negation of itself, then we can introduce $\neg\alpha$. It can be stated as the following axiom schema:

$$\neg\alpha, \text{ where } \alpha \text{ proves } \neg\alpha \text{ [Self-Refutation]}$$

Proof by self-refutation is syntactically a much weaker special case of general proof by contradiction rule. First, it requires that we derive a negation of the assumption, not just any contradiction. Second, it requires that α itself proves its negation, without the aid of any other premises.

I show that the addition of the proof by self-refutation rule to bDL leads to explosiveness.

For any pair of sentences α and β, I derive β from premises α and $\neg\alpha$, using bDL inference rules plus the Self-Refutation axiom schema.

First, I show that $(\neg\alpha \wedge \neg\beta) \wedge (\alpha \vee \beta)$ proves its own negation $\neg((\neg\alpha \vee \neg\beta) \wedge (\alpha \vee \beta))$. Then I use $\neg((\neg\alpha \vee \neg\beta) \wedge (\alpha \vee \beta))$, α, and $\neg\alpha$ to prove β.

Proof 2a: $((\neg\alpha \vee \neg\beta) \wedge (\alpha \vee \beta))$ proves $\neg((\neg\alpha \vee \neg\beta) \wedge (\alpha \vee \beta))$

1	$(\neg\alpha \wedge \neg\beta) \wedge (\alpha \vee \beta)$	Premise
2	$(\neg\alpha \wedge \neg\beta)$	$\wedge$-Elimination, 1
3	$(\alpha \vee \beta)$	$\wedge$-Elimination, 1
4	$(\alpha \vee \beta) \vee (\neg\alpha \wedge \neg\beta)$	Restricted $\vee$-Introduction, 2, 3
5	$(\alpha \vee \beta) \vee \neg(\alpha \vee \beta)$	De Morgan, 4
6	$(\alpha \vee \neg\neg\beta) \vee \neg(\alpha \vee \beta)$	Double negation, 5
7	$(\neg\neg\alpha \vee \neg\neg\beta) \vee \neg(\alpha \vee \beta)$	Double negation, 6
8	$\neg(\neg\alpha \vee \neg\beta) \vee \neg(\alpha \vee \beta)$	De Morgan, 7
9	$\neg((\neg\alpha \vee \neg\beta) \wedge (\alpha \vee \beta))$	De Morgan, 8

[2] Also known as reductio ad absurdum or negation introduction.

Proof 2: Explosion in bDL+[Self-Refutation]

1	α	Premise
2	$\neg\alpha$	Premise
3	$\neg((\neg\alpha \vee \neg\beta) \wedge (\alpha \vee \beta))$	Self-Refutation, Proof 2a
4	$\neg(\neg\alpha \wedge \neg\beta) \vee \neg(\alpha \vee \beta)$	De Morgan, 3
5	$(\neg\neg\alpha \vee \neg\neg\beta) \vee \neg(\alpha \vee \beta)$	De Morgan, 4
6	$(\neg\neg\alpha \vee \beta) \vee \neg(\alpha \vee \beta)$	Double negation, 5
7	$\alpha \vee \beta \vee \neg(\alpha \vee \beta)$	Double negation, 6
8	$\alpha \vee \beta \vee (\neg\alpha \wedge \neg\beta)$	De Morgan, 7
9	$(\alpha \vee \beta \vee \neg\alpha) \wedge (\alpha \vee \beta \vee \neg\beta)$	Distributivity, 8
10	$\alpha \vee \beta \vee \neg\alpha$	$\wedge$-Elimination, 9
11	$\alpha \vee \beta$	Resolution, 10, 1
12	β	Resolution, 11, 2

5 Discussions

Beyond Direct Logic itself, the result have implication for the ongoing research program to identify some "reasonable" and "maximal" subset of classically valid rules and axioms that do not lead to explosiveness.

If we take the boolean equivalences and *wedge*-Elimination for granted, the explosiveness of bDL+[Excluded Middle] essentially rely on only Excluded Middle and Disjunctive Syllogism (a special case of Resolution).

Similarly, the explosiveness of bDL+[Self-Refutation] essentially rely on only Self-Refutation, Disjunctive Syllogism, and Restricted $\vee$-Introduction ($\alpha \vee \beta$ from α and β).

Therefore, the result apply immediately to other paraconsistent logics of the same class. For example, the same proofs (1, 2a, 2) show that Besnard and Hunter's quasi-classical logic [1, 5, 4] also becomes explosive if either Excluded Middle or Sef-Refutation is added.

6 Conclusion

In order to obtain some of the deductive power of classical logic in a paraconsistent logic without explosiveness, some have proposed including the law of excluded middle or proof by self-refutation. However, we see in this paper that the law of excluded middle or the rule of proof by self-refutation in fact leads to explosiveness in Direct Logic [2] and quasi-classical logic.

In response to this finding, Hewitt [2] decided not to include these two proposed rules in Direct Logic. In order to obtain some of the power and convenience of proof by contradiction, Hewitt proposed replacing the rule of proof by self-refutation with the weaker rule of *proof by self-annihilation*, ie., concluding α and $\neg\alpha$ if both $\alpha \vdash \neg\alpha$ and $\neg\alpha \vdash \alpha$. It is an open question whether such a rule would still lead to explosiveness [2].

For a broader perspective, this work contributes to the ongoing research program of identifying some "reasonable" and "maximal" subset of classically valid rules and axioms that do not lead to explosiveness.

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