

Two sources of explosion

Eric Jui-Yi Kao
with Carl Hewitt

Department of Computer Science
Stanford University
Stanford, CA 94305 United States
`erickao@cs.stanford.edu`

August 18, 2011
Inconsistency Robustness 2011

Inconsistency Robustness

- ▶ Automated logical reasoning form a part of many systems.
 - ▶ security policy systems
 - ▶ semantic web
 - ▶ knowledge bases
- ▶ Some logics are explosive
 - I.E. $\{\alpha, \neg\alpha\} \vdash \beta$, for any sentences α, β .
- ▶ Non-explosion is a minimal requirement for inconsistency robustness.

Classical logic is explosive

1	α	Premise
2	$\neg\alpha$	Premise

3	$\neg\beta$	

4	α	Reiteration, 1
5	$\neg\alpha$	Reiteration, 2
6	\perp	Contradiction, 4, 5
7	$\neg\neg\beta$	Proof by contradiction, 3–6
8	β	Double negation elimination, 7

Classical logic is explosive

1	α	Premise
2	$\neg\alpha$	Premise
<hr/>		
3	$\beta \vee \alpha$	\vee -Introduction, 1
4	β	Disjunctive syllogism, 1, 3

Outline

- ▶ Idea: restrict the proof theory of classical logic in some “reasonable” way
 - ▶ Avoid explosion
 - ▶ Retain “maximal” deductive power.
- ▶ Many “design decisions” involved
e.g., cannot retain both \forall -introduction and disjunctive syllogism
- ▶ Direct Logic is one proposal [2]
- ▶ Can we increase its deductive power?
- ▶ We consider two attempts in increasing its deductive power

Boolean Direct Logic rules of inference

Core rules of bDL

$$\frac{\alpha \vee \beta \quad \neg\alpha \vee \psi}{\beta \vee \psi} [\text{Resolution}]$$

$$\frac{\alpha \wedge \beta}{\alpha} [\wedge\text{-Elimination}]$$

$$\frac{\alpha \quad \beta}{\alpha \vee \beta} [\text{Restricted } \vee\text{-Introduction}]$$

$$\frac{\alpha \quad \beta}{\alpha \wedge \beta} [\wedge\text{-Introduction}]$$

Substitution according boolean equivalences

$\frac{\alpha}{s(\alpha)}$ [Substitution],

where $s(\alpha)$ is the result of substituting in α an occurrence of a subformula by an equivalent subformula according to a boolean equivalence below.

Distributivity	$\psi \vee (\alpha \wedge \beta)$	\equiv	$(\psi \vee \alpha) \wedge (\psi \vee \beta)$
	$(\psi \wedge \alpha) \vee (\psi \wedge \beta)$	\equiv	$\psi \wedge (\alpha \vee \beta)$
De Morgan Laws	$\neg(\alpha \wedge \beta)$	\equiv	$\neg\alpha \vee \neg\beta$
	$\neg(\alpha \vee \beta)$	\equiv	$\neg\alpha \wedge \neg\beta$
Double negation	$\neg\neg\alpha$	\equiv	α
Idempotence	$\alpha \vee \alpha$	\equiv	α
	$\alpha \wedge \alpha$	\equiv	α

Some properties of bDL

- ▶ bDL is not explosive
- ▶ bDL is “reasonable”
- ▶ Can we make bDL more powerful?
e.g., bDL cannot prove $p \vee \neg p$

Law of excluded middle

- ▶ Intuitively: no sentence can be neither true nor false.
- ▶ Axiom schema

$$\alpha \vee \neg\alpha \text{ [Excluded Middle]}$$

- ▶ Not obvious whether $\text{bDL} + [\text{Excluded Middle}]$ is explosive

E.G. bDL plus the axioms $\{p \vee \neg p : p \in \text{PROPOSITIONS}\}$ is not explosive [5].

Excluded Middle is explosive

1	α	Premise
2	$\neg\alpha$	Premise
3	$(\alpha \wedge \neg\beta) \vee \neg(\alpha \wedge \neg\beta)$	Excluded Middle
4	$(\alpha \wedge \neg\beta) \vee \neg\alpha \vee \neg\neg\beta$	De Morgan, 3
5	$(\alpha \wedge \neg\beta) \vee \neg\alpha \vee \beta$	Double negation, 4
6	$(\alpha \vee \neg\alpha \vee \beta) \wedge (\neg\beta \vee \neg\alpha \vee \beta)$	Distributivity, 5
7	$\alpha \vee \neg\alpha \vee \beta$	\wedge -Elimination, 6
8	$\alpha \vee \beta$	Resolution, 7, 1
9	β	Resolution, 8, 2

Proof by self-refutation

- ▶ Intuitively: If a sentence negates itself, it must be false.
- ▶ If a sentence α derives the negation of itself, then we can introduce $\neg\alpha$.
- ▶ Axiom schema:

$\neg\alpha$, where α proves $\neg\alpha$ [Self-Refutation]

Proof 2a: $((\neg\alpha \vee \neg\beta) \wedge (\alpha \vee \beta))$ proves $\neg((\neg\alpha \vee \neg\beta) \wedge (\alpha \vee \beta))$

1	$(\neg\alpha \wedge \neg\beta) \wedge (\alpha \vee \beta)$	Premise
2	$(\neg\alpha \wedge \neg\beta)$	\wedge -Elimination, 1
3	$(\alpha \vee \beta)$	\wedge -Elimination, 1
4	$(\alpha \vee \beta) \vee (\neg\alpha \wedge \neg\beta)$	Restricted \vee -Introduction, 2, 3
5	$(\alpha \vee \beta) \vee \neg(\alpha \vee \beta)$	De Morgan, 4
6	$(\alpha \vee \neg\neg\beta) \vee \neg(\alpha \vee \beta)$	Double negation, 5
7	$(\neg\neg\alpha \vee \neg\neg\beta) \vee \neg(\alpha \vee \beta)$	Double negation, 6
8	$\neg(\neg\alpha \vee \neg\beta) \vee \neg(\alpha \vee \beta)$	De Morgan, 7
9	$\neg((\neg\alpha \vee \neg\beta) \wedge (\alpha \vee \beta))$	De Morgan, 8

1	α	Premise
2	$\neg\alpha$	Premise
3	$\neg((\neg\alpha \vee \neg\beta) \wedge (\alpha \vee \beta))$	Self-Refutation, Proof 2a
4	$\neg(\neg\alpha \wedge \neg\beta) \vee \neg(\alpha \vee \beta)$	De Morgan, 3
5	$(\neg\neg\alpha \vee \neg\neg\beta) \vee \neg(\alpha \vee \beta)$	De Morgan, 4
6	$(\neg\neg\alpha \vee \beta) \vee \neg(\alpha \vee \beta)$	Double negation, 5
7	$\alpha \vee \beta \vee \neg(\alpha \vee \beta)$	Double negation, 6
8	$\alpha \vee \beta \vee (\neg\alpha \wedge \neg\beta)$	De Morgan, 7
9	$(\alpha \vee \beta \vee \neg\alpha) \wedge (\alpha \vee \beta \vee \neg\beta)$	Distributivity, 8
10	$\alpha \vee \beta \vee \neg\alpha$	\wedge -Elimination, 9
11	$\alpha \vee \beta$	Resolution, 10, 1
12	β	Resolution, 11, 2

Design decisions

- ▶ Let's take the boolean equivalences and \wedge -Elimination for granted
- ▶ The explosiveness of $\text{bDL} + [\text{Excluded Middle}]$ essentially rely on only
 - ▶ Excluded Middle and
 - ▶ Disjunctive Syllogism (a special case of Resolution)
- ▶ Direct Logic chooses Disjunctive Syllogism and leaves out Excluded Middle
- ▶ The explosiveness of $\text{bDL} + [\text{Self-Refutation}]$ essentially rely on only
 - ▶ Self-Refutation,
 - ▶ Disjunctive Syllogism, and
 - ▶ Restricted \vee -Introduction ($\alpha \vee \beta$ from α and β)
- ▶ Direct Logic replaces Self-Refutation with a weaker rule.

Other logics

- ▶ The results apply to other paraconsistent logics that support the rules used.
- ▶ For example, Besnard and Hunter's quasi-classical logic [1, 4, 3] also becomes explosive if either Excluded Middle or Self-Refutation is added.






Open questions

Consider the set of valid inference rules in classical boolean logic:

$$\mathcal{R} = \left\{ \frac{\Phi_1 \cdots \Phi_n}{\Psi} : \Phi_1 \cdots \Phi_n \models \Psi \right\}$$

for any instances $\phi_1, \dots, \phi_n, \psi$ of $\Phi_1, \dots, \Phi_n, \Psi$

- ▶ Find a maximal subset S of \mathcal{R} such that the logic induced by S is not explosive.
- ▶ Can the induced logic be axiomatized by a finite number of inference rules?
- ▶ Is the induced logic decidable?
- ▶ Characterize the space of all such $S \subseteq \mathcal{R}$

-  Besnard, P., Hunter, A.: Quasi-classical logic: Non-trivializable classical reasoning from inconsistent information. In: Proceedings of the European Conference on Symbolic and Quantitative Approaches to Reasoning and Uncertainty. pp. 44–51. Springer-Verlag, London, UK (1995), <http://portal.acm.org/citation.cfm?id=646561.695561>
-  Hewitt, C.: Common sense for inconsistency robust information integration using direct logic reasoning and the actor model. arXiv CoRR abs/0812.4852 (2011)
-  Hunter, A.: Paraconsistent logics. In: Handbook of Defeasible Reasoning and Uncertain Information. pp. 11–36. Kluwer (1996)
-  Hunter, A.: Reasoning with contradictory information using quasi-classical logic. Journal of Logic and Computation 10, 677–703 (1999)
-  Kao, E.J.Y., Genesereth, M.: Achieving cut, deduction, and other properties with a variation on quasi-classical logic (2011),

[http://dl.dropbox.com/u/5152476/working-papers/
modified-quasiclassical/main.pdf](http://dl.dropbox.com/u/5152476/working-papers/modified-quasiclassical/main.pdf), working paper

Two sources of explosion

Eric Jui-Yi Kao
with Carl Hewitt

Department of Computer Science
Stanford University
Stanford, CA 94305 United States
`erickao@cs.stanford.edu`

August 18, 2011
Inconsistency Robustness 2011

Proof by contradiction

- ▶ If by assuming a sentence α we derive a contradiction, then we can conclude $\neg\alpha$.
- ▶ It can be stated as the following meta-rule:
If $\Sigma, \alpha \vdash \psi$ and $\Sigma, \alpha \vdash \neg\psi$, then conclude $\Sigma \vdash \neg\alpha$.
- ▶ Proof by contradiction easily leads to explosiveness. For any sentences α and β ,

$$\{\alpha, \neg\alpha\}, \neg\beta \vdash \alpha \text{ and } \{\alpha, \neg\alpha\}, \neg\beta \vdash \neg\alpha,$$

hence $\{\alpha, \neg\alpha\} \vdash \neg\neg\beta$ using proof by contradiction.

Self-refutation is explosive

I show that the addition of the proof by self-refutation rule to bDL leads to explosiveness.

For any pair of sentences α and β , I derive β from premises α and $\neg\alpha$, using bDL inference rules plus the Self-Refutation axiom schema.

First, I show that $(\neg\alpha \wedge \neg\beta) \wedge (\alpha \vee \beta)$ proves its own negation $\neg((\neg\alpha \vee \neg\beta) \wedge (\alpha \vee \beta))$. Then I use $\neg((\neg\alpha \vee \neg\beta) \wedge (\alpha \vee \beta))$, α , and $\neg\alpha$ to prove β .