Introduction to Logic

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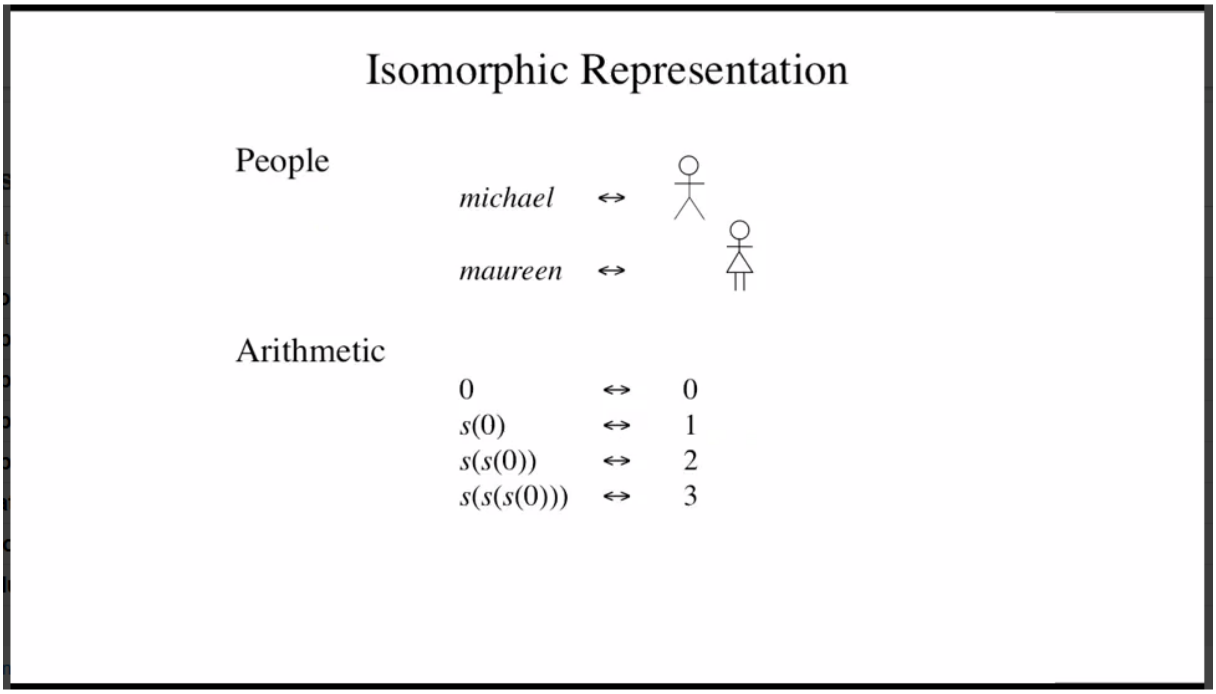
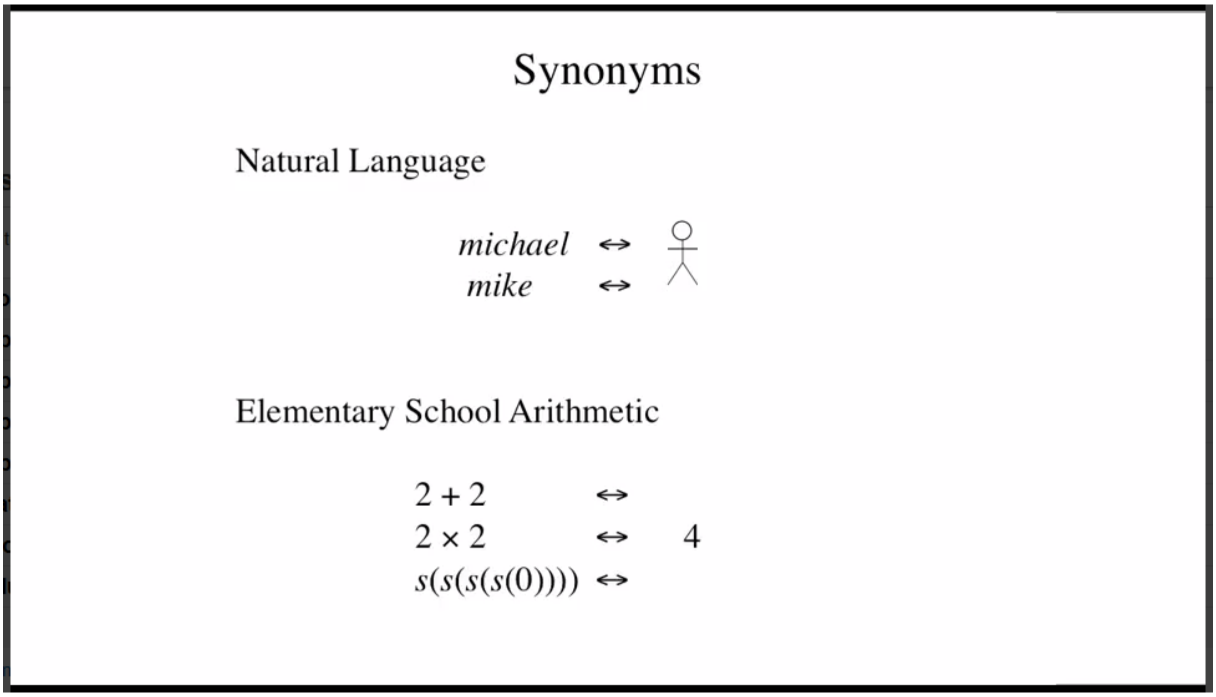
Equality


Isomorphic Representation
People
michael ↔
maureen ↔
Arithmetic
0 ↔ 0
s(0) ↔ 1
s(s(0)) ↔ 2
s(s(s(0))) ↔ 3


Synonyms
Natural Language
michael ↔
mike ↔
Elementary School Arithmetic
2 + 2 ↔
2 × 2 ↔ 4
s(s(s(s(0)))) ↔

Equality

An equation *equal*(σ, τ) is true if and only if the terms in the equation *refer to the same object* in the world.

$$equal(f(a),f(b))$$

Shorthand for equations:

$$f(a) = f(b)$$

Equality Axioms

Equality Axioms

Reflexivity

$$\forall x.(x = x)$$

Symmetry:

$$\forall x.\forall y.(x = y \Rightarrow y = x)$$

Transitivity:

$$\forall x.\forall y.\forall z.(x = y \wedge y = z \Rightarrow x = z)$$

Equality Problem

Given premises $(b = a)$ and $(b = c)$, prove $(a = c)$.

Equality Proof

1.	$b = a$	Premise
2.	$b = c$	Premise
3.	$\forall x.(x = x)$	Premise
4.	$\forall x.\forall y.(x = y \Rightarrow y = x)$	Premise
5.	$\forall x.\forall y.\forall z.(x = y \wedge y = z \Rightarrow x = z)$	Premise
6.	$b = a \Rightarrow a = b$	2 x UE: 4
7.	$a = b$	IE: 6, 1
8.	$a = b \wedge b = c \Rightarrow a = c$	3 x UE: 5
9.	$a = b \wedge b = c$	AI: 7, 2
10.	$a = c$	IE: 8, 9

Substitution

Equality Problem

1.	$f(a) = b$	Premise
2.	$f(b) = a$	Premise
3.	$\forall x.(x = x)$	Premise
4.	$\forall x.\forall y.(x = y \Rightarrow y = x)$	Premise
5.	$\forall x.\forall y.\forall z.(x = y \wedge y = z \Rightarrow x = z)$	Premise
…	…	…
m.	$b = f(a)$	
n.	$a = f(b)$	
…	…	…
o.	$f(a)=y \wedge y = b$	

Substitution Axioms for Functions

Unary Function:

$$\forall x.\forall y.\forall z.(f(x)=z \wedge x=y \Rightarrow f(y)=z)$$

Binary Function:

$$\forall u.\forall v.\forall x.\forall y.\forall z.(f(u,v)=z \wedge u=x \wedge v=y \Rightarrow f(x,y)=z)$$

Proof With Substitution Axiom

1.	$f(a) = b$	Premise
2.	$f(b) = a$	Premise
3.	$\forall x.(x = x)$	Premise
4.	$\forall x.\forall y.(x = y \Rightarrow y = x)$	Premise
5.	$\forall x.\forall y.\forall z.(x = y \wedge y = z \Rightarrow x = z)$	Premise
6.	$\forall x.\forall y.\forall z.(f(x)=z \wedge x=y \Rightarrow f(y)=z)$	Premise
7.	$f(b)=a \wedge b=f(a) \Rightarrow f(f(a))=a$	3 x UE: 6
8.	$f(a)=b \Rightarrow b=f(a)$	2 x UE: 4
9.	$b=f(a)$	IE: 8, 1
10.	$f(b) = a \wedge b=f(a)$	AI: 2, 9
11.	$f(f(a))=a$	IE: 7, 10

Substitution Axioms for Relations

Substitution axioms for relation constants too.

$$\forall x.\forall y.(x=y \wedge p(x) \Rightarrow p(y))$$

Substitution axioms for multiple arguments

$$\forall u.\forall v.\forall x.\forall y.(u=x \wedge v=y \wedge p(u,v) \Rightarrow p(x,y))$$

Note that we need one substitution for each function constant and each relation constant.

Equality in Our Proof System

Equality Introduction (QI)

$\tau = \tau$

where τ is any term

Examples

$$a = a$$

$$f(a) = f(a)$$

$$f(x) = f(x)$$

Equality Elimination (QE)

$$\frac{\begin{array}{l}\tau_1 = \tau_2 \\ \varphi[\tau_1]\end{array}}{\varphi[\tau_2]}$$

where τ_2 is substitutable for τ_1 in φ

Examples

Premises:

$x = b$

$hates(x,x)$

Conclusions:

$hates(x,b)$

$hates(b,x)$

$hates(b,b)$

Equality Problem

Given premises $(b = a)$ and $(b = c)$, prove $(a = c)$.

Equality Proof

1.	$b = a$	Premise
2.	$b = c$	Premise
3.	$\forall x.(x = x)$	Premise
4.	$\forall x.\forall y.(x = y \Rightarrow y = x)$	Premise
5.	$\forall x.\forall y.\forall z.(x = y \wedge y = z \Rightarrow x = z)$	Premise
6.	$b = a \Rightarrow a = b$	2 x UE: 4
7.	$a = b$	IE: 6, 1
8.	$a = b \wedge b = c \Rightarrow a = c$	3 x UE: 5
9.	$a = b \wedge b = c$	AI: 7, 2
10.	$a = c$	IE: 8, 9

Equality Proof

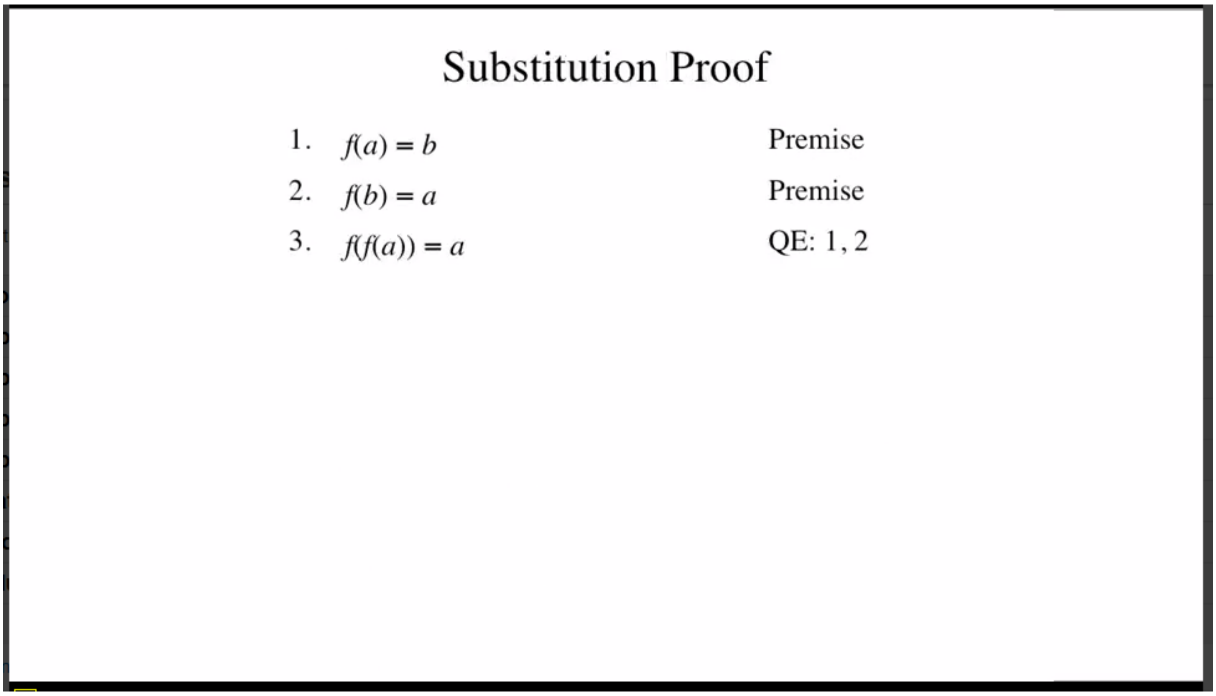
1.	$b = a$	Premise
2.	$b = c$	Premise
3.	$a = c$	QE: 1, 2

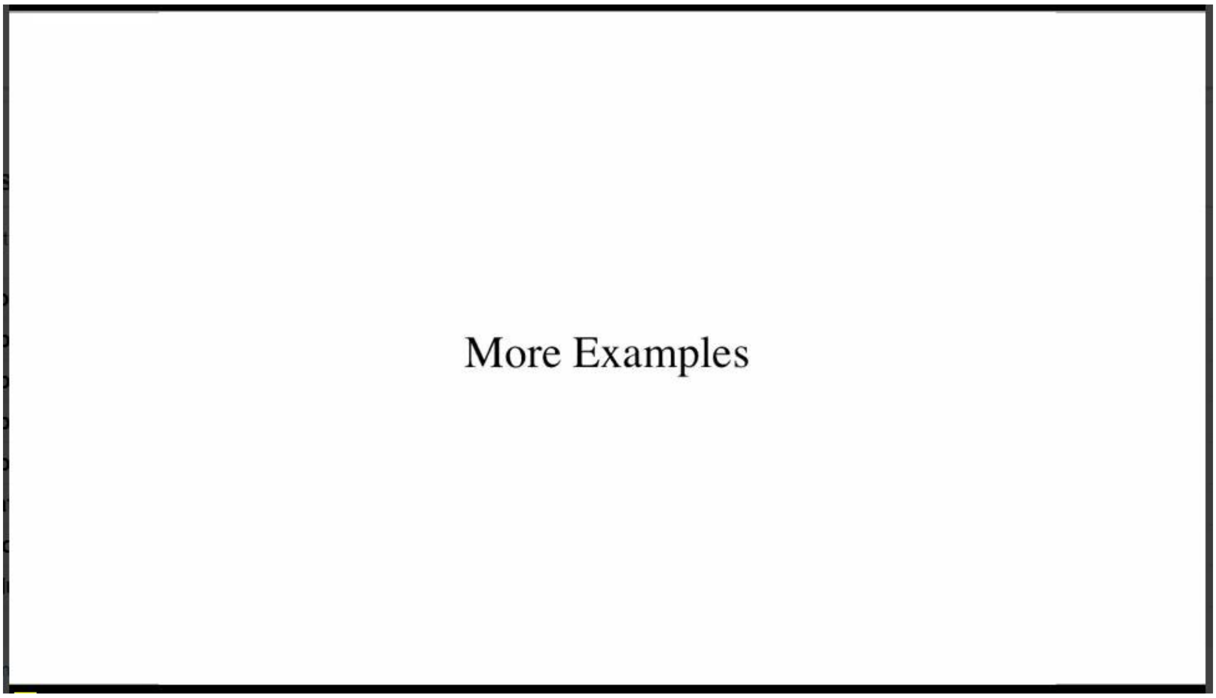
Substitution Problem

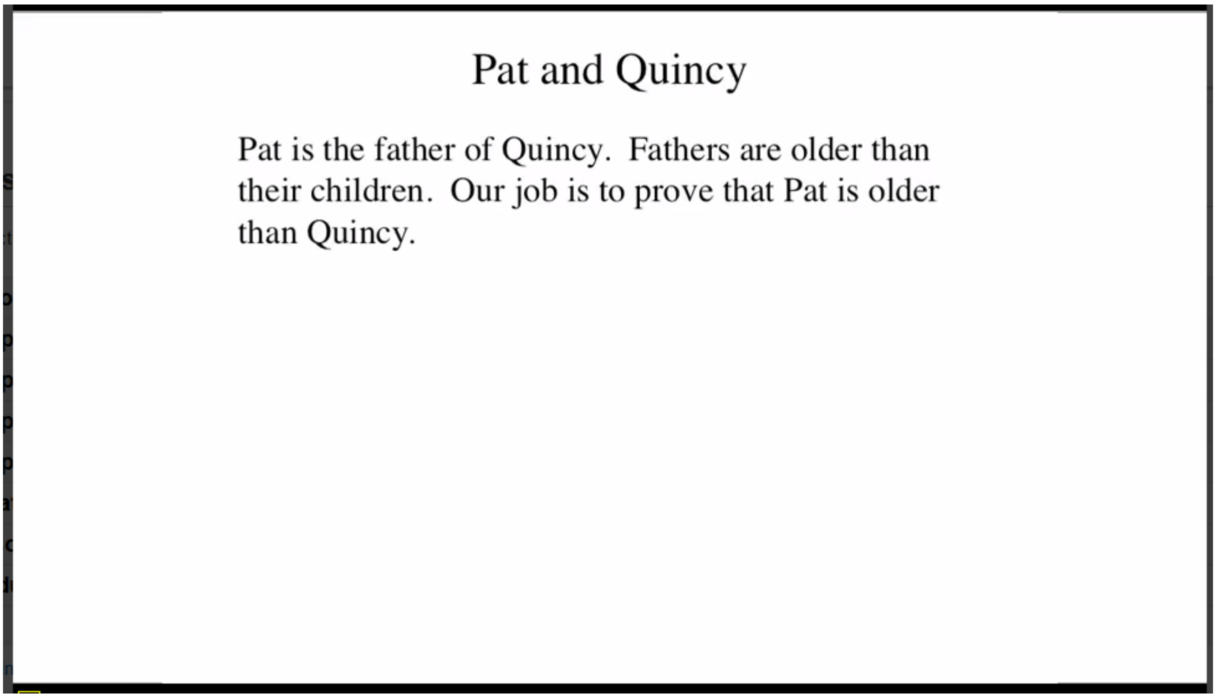
1.	$f(a) = b$	Premise
2.	$f(b) = a$	Premise

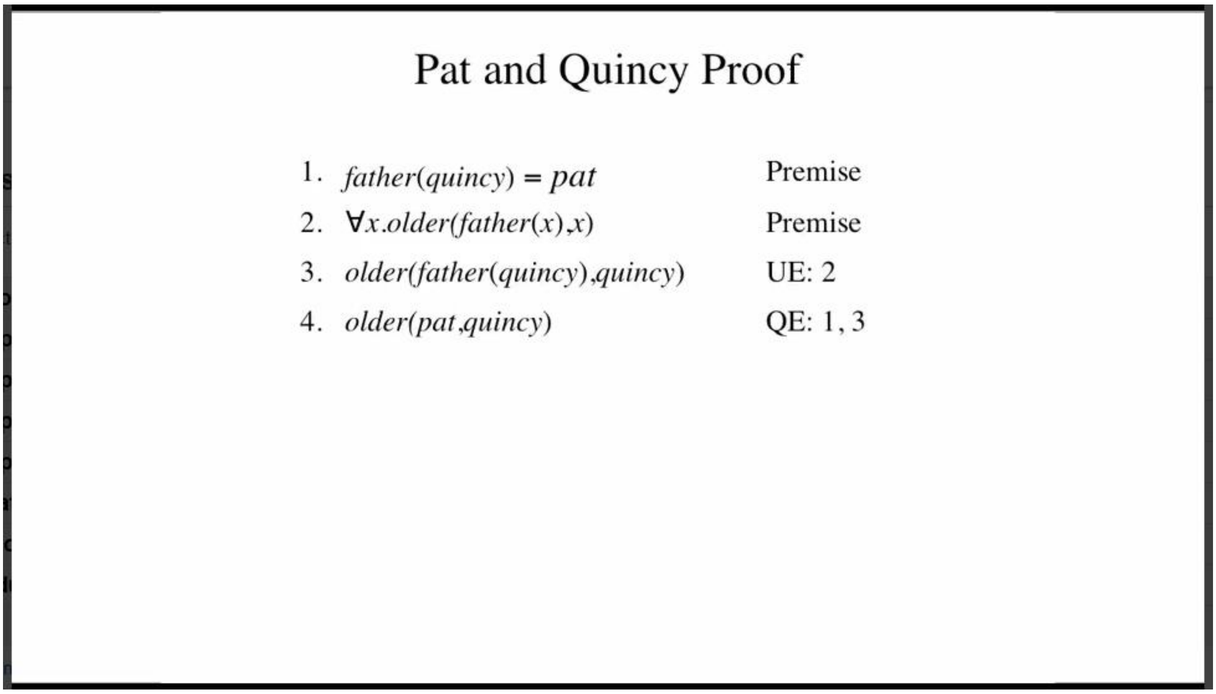
Proof With Substitution Axiom

1.	$f(a) = b$	Premise
2.	$f(b) = a$	Premise
3.	$\forall x.(x = x)$	Premise
4.	$\forall x.\forall y.(x = y \Rightarrow y = x)$	Premise
5.	$\forall x.\forall y.\forall z.(x = y \wedge y = z \Rightarrow x = z)$	Premise
6.	$\forall x.\forall y.\forall z.(f(x)=z \wedge x=y \Rightarrow f(y)=z)$	Premise
7.	$f(b)=a \wedge b=f(a) \Rightarrow f(f(a))=a$	3 x UE: 6
8.	$f(a)=b \Rightarrow b=f(a)$	2 x UE: 4
9.	$b=f(a)$	IE: 8, 1
10.	$f(b) = a \wedge b=f(a)$	AI: 2, 9
11.	$f(f(a))=a$	IE: 7, 10


Substitution Proof
1. f(a) = b Premise
2. f(b) = a Premise
3. f(f(a)) = a QE: 1, 2


More Examples


Pat and Quincy
Pat is the father of Quincy. Fathers are older than their children. Our job is to prove that Pat is older than Quincy.


Pat and Quincy Proof
1. father(quincy) = pat
Premise
2. ∀x.older(father(x),x)
Premise
3. older(father(quincy),quincy)
UE: 2
4. older(pat,quincy)
QE: 1, 3

Disjunction Problem

Suppose we know $p(a)$ and $p(b)$; and suppose we know that $(a{=}c \lor b{=}c)$. Our job is to prove $p(c)$.

Disjunction Proof

1.	$p(a)$	Premise
2.	$p(b)$	Premise
3.	$a{=}c \lor b{=}c$	Premise
4.	$\quad$ | $a{=}c$	Assumption
5.	$\quad$ | $p(c)$	QE: 4, 1
6.	$a{=}c \Rightarrow p(c)$	II: 4, 5
7.	$\quad$ | $b{=}c$	Assumption
8.	$\quad$ | $p(c)$	QE: 7, 2
9.	$b{=}c \Rightarrow p(c)$	II: 7, 8
10.	$p(c)$	EE: 3, 6, 8