Introduction to Logic
First-Order Logic

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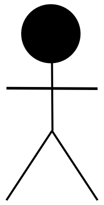
Objects with Multiple Names

People (Nicknames):

$$michael \leftrightarrow$$
$$mike \leftrightarrow$$



Arithmetic:

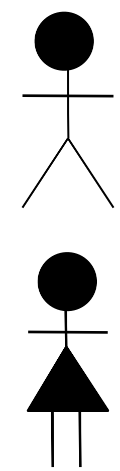
$$
\begin{array}{lll}
plus(s(0), s(0)) & \leftrightarrow & \\
times(s(s(0)), s(0)) & \leftrightarrow & 2 \\
s(s(0)) & \leftrightarrow &
\end{array}
$$

Objects with No Names

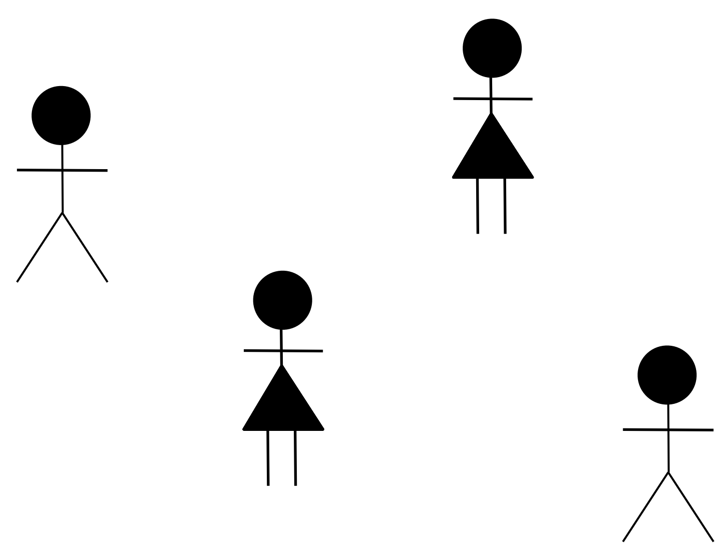
People

michael $\leftrightarrow$

maureen $\leftrightarrow$



Unnamed People



Mathematical Objects Without Names

Named real numbers (countably many):

123

34.12

pi

e

All the others (uncountably many):

3.141544878723489184093893477809489084...

6.878989783975975738975379875837593358...

How many floating point numbers are there?

First Order Logic

Syntax

Identical to Herbrand Logic

Semantics

Herbrand Logic - defined in terms of language

First Order Logic - defined in terms of external worlds

Language of First Order Logic

$$p(a) \qquad q(b,c)$$

$$\neg p(b) \qquad \forall x.(p(x) \Rightarrow q(x,f(x)))$$

$$p(c) \vee p(d) \qquad \exists x.q(x,d)$$

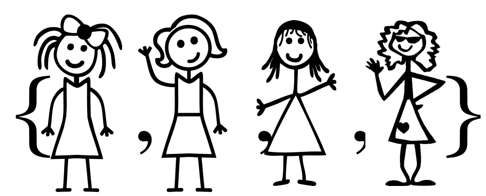
Identical to Syntax of Functional Logic

Interpretations

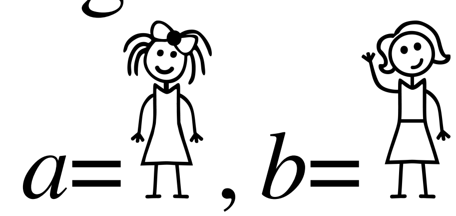
A vocabulary is a set of symbols.

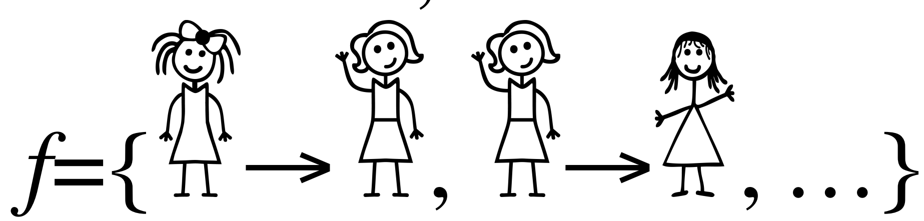
$$\{a, b, f, q\}$$

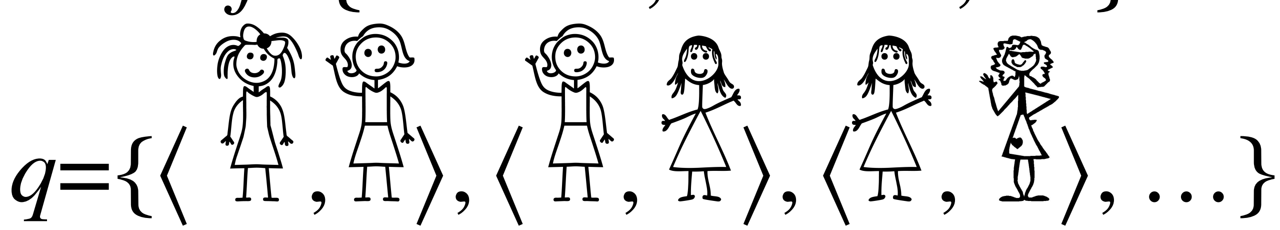
A *universe of discourse* is an *arbitrary* set of objects.



An *interpretation* is an assignment to symbols in language.







Interpretations

A vocabulary is a set of symbols.

$$\{a, b, f, q\}$$

A *universe of discourse* is an *arbitrary* set of objects.

$$\{1, 2, 3, 4\}$$

An *interpretation* is an assignment to symbols in language.

$$a{=}1,\ b{=}2$$
$$f{=}\{1 \rightarrow 2,\ 2 \rightarrow 3,\ \ldots\}$$
$$q{=}\{\langle 1{,}2\rangle{,}\langle 2{,}3\rangle{,}\langle 2{,}4\rangle,\ \ldots\}$$

Entailment defined in terms of ***all*** *interpretations over*
all *possible universes of* ***all*** *possible sizes*

Variable Assignments

A *variable assignment* for a universe of discourse U is a function assigning variables to objects in U.

$$v\text{: } \mathit{Variable} \rightarrow U$$

Universe of Discourse:

$U = \{1, 2, 3\}$

Example:

$v(x) = 1$

$v(y) = 2$

$v(z) = 3$

Example:

$v(x) = 2$

$v(y) = 2$

$v(z) = 2$

Value Assignments

A *value assignment* s_{iv} based on interpretation i and variable assignment v is a mapping from the terms of the language into the universe of discourse.

$$s_{iv}(\sigma)=i(\sigma)$$

$$s_{iv}(\upsilon)=v(\upsilon)$$

$$s_{iv}(\pi(\tau_1,\ldots,\tau_n))=i(\pi)(s_{iv}(\tau_1),\ldots,s_{iv}(\tau_n))$$

Relational Sentences

A truth assignment satisfies a relational sentence if and only if the tuple of objects denoted by the arguments is a member of the relation denoted by the relation constant.

$$t_{iv}(\rho(\tau_1,\ldots,\tau_n)) = true \;\text{ if } \langle s_{iv}(\tau_1),\ldots, s_{iv}(\tau_1)\rangle \in i(\rho)$$
$$= false \text{ otherwise}$$

Logical Sentences

$t_{iv}(\neg\varphi) = \mathit{true}$ iff $t_{iv}(\varphi) = \mathit{false}$

$t_{iv}(\varphi \wedge \psi) = \mathit{true}$ iff $t_{iv}(\varphi) = \mathit{true}$ and $t_{iv}(\psi) = \mathit{true}$

$t_{iv}(\varphi \vee \psi) = \mathit{true}$ iff $t_{iv}(\varphi) = \mathit{true}$ or $t_{iv}(\psi) = \mathit{true}$

$t_{iv}(\varphi \Rightarrow \psi) = \mathit{true}$ iff $t_{iv}(\varphi) = \mathit{false}$ or $t_{iv}(\psi) = \mathit{true}$

$t_{iv}(\varphi \Leftrightarrow \psi) = \mathit{true}$ iff $t_{iv}(\varphi) = t_{iv}(\psi)$

Quantified Sentences

Intuitively, a universally quantified sentence is true if and only if it is true no matter what *value* we assign to the universally quantified variable.

Intuitively, an existentially quantified sentence is true if and only if it is true for some *value* of the existentially quantified variable.

Stating these definitions precisely is a little tricky due to the possibility of nested quantifiers.

$$\forall x.(\exists y.r(x,y) \Rightarrow \forall x.r(x,x))$$

Versions

A *version* $v[\omega\leftarrow x]$ of a variable assignment v is the variable assignment that agrees with v on all variables except ω, which is assigned the value x.

$$
\begin{array}{ll}
v[\omega\leftarrow x](\theta) = x & \textit{if } \theta=\omega \\
v[\omega\leftarrow x](\theta) = v(\theta) & \textit{if } \theta\neq\omega
\end{array}
$$

Quantified Sentences

A universally quantified sentence is true in interpretation *i* and variable assignment *v* if and only if the scope is true for *i* and *every* version of *v*.

$$t_{iv}(\forall\omega.\varphi\ )=\mathit{true} \text{ iff } t_{iv[\omega\leftarrow x]}(\varphi)=\mathit{true} \text{ for all } x\in|i|.$$

An existentially quantified sentence is true in interpretation *i* and variable assignment *v* if and only if the scope is true for *i* and *some* version of *v*.

$$t_{iv}(\exists\omega.\varphi\ )=\mathit{true} \text{ iff } t_{iv[\omega\leftarrow x]}(\varphi)=\mathit{true} \text{ for some } x\in|i|.$$

Comparison

First Order Logic: A universally quantified sentence is true in interpretation *i* and variable assignment *v* if and only if the scope is true for *i* and *every* version of *v* compatible with interpretation *i*.

Herbrand Logic: A universally quantified sentence is true in a truth assignment if and only if every instance is true.

Herbrand Logic vs First-Order Logic

In Herbrand Logic, if $\Delta \models p(\tau)$ for *every* ground term τ, does $\Delta \models \forall x.p(x)$?

Yes.

Herbrand Logic vs First-Order Logic

In Herbrand Logic, if $\Delta \models p(\tau)$ for *every* ground term τ, does $\Delta \models \forall x.p(x)$?

Yes.

In First-Order Logic, if $\Delta \models p(\tau)$ for *every* ground term τ, does $\Delta \models \forall x.p(x)$?

No. There can be objects without names.

Herbrand Logic and Uncountable Worlds

Can we describe uncountable world in Herbrand Logic?

No. There only countably many terms and countably many ground sentences in our language.

Upshot: It is not possible to axiomatize uncountable worlds in Herbrand Logic.

FOL and Uncountable Worlds

Can we describe uncountable worlds in First-Order Logic?

Yes. There can be objects without names.

Lowenhein-Skolem Theorem: If a set of sentences in First Order Logic has a model of one infinite cardinality, then it has a model of every infinite cardinality. (This striking result is true but not obvious.)

Upshot: It is not possible to write a sentence in First Order Logic that is true in an uncountable world and not true in any countably infinite world or vice-versa.

Completeness of Herbrand Logic

Peano Arithmetic

Transitive Closure

Both of these are finite axiomatizations and are complete (i.e. they precisely define which sentences are true and which are false). There are no *non-standard* models.

Incompleteness of First-Order Logic

First-Order Logic (FOL) theories with infinite universes have *nonstandard models* (unintended models that *cannot be excluded*).

Upshot: FOL is *weaker* than Herbrand Logic. Some notions that can be defined exactly in Herbrand Logic cannot be defined in FOL without allowing nonstandard models, e.g. Peano Arithmetic, transitive closure.

Inferential Completeness of First Order Logic

There **is** a proof procedure for First Order Logic that is both sound and complete. (Spoiler Alert: Fitch without Domain Closure and Induction does the trick.)

Moreover, by systematically applying the procedure, possible to compute all logical consequences of any enumerable set of premises. (Apply Fitch to produce all finite proofs in systematic way.)

Upshot: provability and logical entailment are semi-decidable (though not decidable).

Inferential Incompleteness of Herbrand Logic

The axiomatization of Peano Arithmetic in Herbrand Logic completely defines Peano Arithmetic.

If Herbrand entailment were semi-decidable, the set of all true sentences would be enumerable.

Godel's incompleteness theorem tells us that the set of all true sentences of Peano arithmetic is not computably enumerable.

Consequently, there is no complete (semi-decidable) proof procedure for Herbrand Logic.

Comparison

Theorem: Any sound proof procedure for First Order Logic is sound for Herbrand Logic.

Even though there is no complete proof procedure, Herbrand logic is *not* weaker. In fact, Herbrand logic is ***stronger*** than FOL. There are simply *more* things that are true.

We cannot prove them all, but we can prove everything we could prove in First Order Logic; and, by building in induction, we can prove ***more*** things.

Summary

First Order Logic:

- Compact
- Complete proof procedure
- Semi-decidable

Herbrand Logic:

- Not compact
- No complete proof procedure
- Not even semi-decidable

Comparison of Herbrand Logic over First Order Logic:

- Herbrand Semantics simpler and more intuitive
- FOL can be used to describe uncountable worlds
- More things definable in Herbrand Logic
- Greater inferential power in HL but not complete

Herbrand Manifesto

http://logic.stanford.edu/herbrand/manifesto.html