Boring lecture

Introduction to Logic
Direct Proofs

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Logical Entailment

A set of premises Δ logically entails a conclusion φ (written $\Delta \models \varphi$) if and only if every interpretation that satisfies Δ also satisfies φ.

Semantic Reasoning (Truth Tables)

Premises				Conclusion		
m	*p*	*q*		*m*	*p*	*q*
T	T	T		T	T	T
T	T	F	→	T	T	F
T	F	T		T	F	T
T	F	F	→	T	F	F
F	T	T		F	T	T
F	T	F		F	T	F
F	F	T		F	F	T
F	F	F		F	F	F

With n constants, there are 2^n truth assignments.

Symbolic Manipulation (Metatheorems)

Is the sentence $((p \Leftrightarrow \neg q) \Rightarrow (\neg p \mid \neg q))$ valid, contingent, or unsatisfiable?

$$(p \Leftrightarrow \neg q) \Rightarrow (\neg p \mid \neg q)$$

$$(p \Leftrightarrow \neg q) \Rightarrow (p \Rightarrow \neg q)$$

$$(p \Rightarrow \neg q) \wedge (\neg q \Rightarrow p) \Rightarrow (p \Rightarrow \neg q)$$

$$(p \Rightarrow \neg q) \wedge (\neg q \Rightarrow p) \models (p \Rightarrow \neg q)$$

$$\{(p \Rightarrow \neg q), (\neg q \Rightarrow p)\} \models (p \Rightarrow \neg q)$$

$$\{(p \Rightarrow \neg q)\} \models (p \Rightarrow \neg q)$$

Monotonicity Theorem

Symbolic Manipulation (Proofs)

Truth Tables:
- Guaranteed to succeed
- Often quite large

Problem Rewriting using metatheorems:
- Less work than truth tables
- More intuitive than truth tables
- *Sometimes works, sometimes don't*

Theorem Proving:
- Less work than truth tables
- More intuitive than truth tables
- *Theorem proving always works. Systematic.*

Proof Systems

Popular Types of Proof Systems:

→ Direct Proofs (Hilbert)

Natural Deduction (Fitch)

Refutation proofs (Resolution)

Others:

Gentzen Systems

Sequent Calculi

and so forth

Direct Proofs

Schemas / Schemata

A *pattern* / *schema* is an expression satisfying the grammatical rules of our language except for the occurrence of *metavariables* (written here as Greek letters) in place of various subparts of the expression.

For example, the following expression is a schema with metavariables ϕ and ψ.

$$\phi \Rightarrow (\psi \Rightarrow \phi)$$

Instances

An *instance* of a schema is a sentence obtained by consistently substituting sentences for the metavariables in the rule.

Schema:

$$\phi \Rightarrow (\psi \Rightarrow \phi)$$

Instances:

$$p \Rightarrow (q \Rightarrow p)$$

$$\neg p \Rightarrow (q \Rightarrow \neg p)$$

$$(p \Rightarrow q) \Rightarrow ((q \Rightarrow r) \Rightarrow (p \Rightarrow q))$$

Non-Instance:

$$p \Rightarrow (q \Rightarrow r)$$

Valid Schemata

A *valid schema* is a sentence pattern denoting an infinite set of sentences, all of which are valid.

Examples:

Reflexivity: $(\varphi \Rightarrow \varphi)$

Negation Elimination: $(\neg\neg\varphi \Rightarrow \varphi)$

Negation Introduction: $(\varphi \Rightarrow \neg\neg\varphi)$

Disjunctive Tautology: $(\varphi \vee \neg\varphi)$

Non-Examples:

$(\varphi \wedge \neg\varphi)$

$(\varphi \Rightarrow \neg\varphi)$

Rules of Inference

A *rule of inference* is a pattern of reasoning consisting of zero or more schemas, called *premises*, and one or more additional schemas, called *conclusions*.

$$\frac{\begin{array}{l}\phi \Rightarrow \psi \\ \phi\end{array}}{\psi}$$

This rule is *Implication Elimination* or *Modus Ponens*.

Rules of Inference for ¬ and ⇒

Implication Elimination

$$\frac{\phi \Rightarrow \psi \qquad \phi}{\psi}$$

Implication Creation

$$\frac{\psi}{\phi \Rightarrow \psi}$$

Implication Distribution

$$\frac{(\phi \Rightarrow (\psi \Rightarrow \chi))}{(\phi \Rightarrow \psi) \Rightarrow (\phi \Rightarrow \chi)}$$

Implication Reversal

$$\frac{\neg\psi \Rightarrow \neg\phi}{\phi \Rightarrow \psi}$$

Rule Instances

An *instance* of a rule of inference is the rule obtained by consistently substituting sentences for the metavariables in the rule.

Rule of Inference:

$$\begin{array}{l} \phi \Rightarrow \psi \\ \phi \\ \hline \psi \end{array}$$

Instances:

$$\begin{array}{l} p \Rightarrow q \\ p \\ \hline q \end{array} \qquad \begin{array}{l} \neg p \Rightarrow q \\ \neg p \\ \hline q \end{array} \qquad \begin{array}{l} (p \Rightarrow q) \Rightarrow (q \Rightarrow r) \\ p \Rightarrow q \\ \hline q \Rightarrow r \end{array}$$

Rule Application

Premises:

p

$(p \Rightarrow q)$

$(p \Rightarrow q) \Rightarrow (q \Rightarrow r)$

Rule of Inference:

$$\frac{\phi \Rightarrow \psi \qquad \phi}{\psi}$$

Rule Instance 1:

$$\frac{p \Rightarrow q \qquad p}{q}$$

Rule Instance 2:

$$\frac{(p \Rightarrow q) \Rightarrow (q \Rightarrow r) \qquad (p \Rightarrow q)}{(q \Rightarrow r)}$$

Conclusion 1:

q

Conclusion 2:

$(q \Rightarrow r)$

Proof

A *direct proof* of a conclusion from a set of premises is a sequence of sentences terminating in the conclusion in which each item is either

(1) a premise,

(2) an instance of a valid schema, or

(3) the result of applying a rule of inference to earlier items in sequence.

Top Level Sentences Only!!!

Premises:

$(p \Rightarrow q)$

$(p \Rightarrow r)$

Rule of Inference:

$$\frac{\begin{array}{l}\phi \Rightarrow \psi \\ \phi\end{array}}{\psi}$$

Rule Instance:

$$\frac{\begin{array}{l}(p \Rightarrow q) \\ p\end{array}}{q}$$

Conclusion:

X $(q \Rightarrow r)$ X No, no, no!

Rules cannot be applied to parts of sentences!!!

Example 1

Given p and $(p \Rightarrow q)$ and $((p \Rightarrow q) \Rightarrow (q \Rightarrow r))$, prove r.

Example 1

Given p and $(p \Rightarrow q)$ and $((p \Rightarrow q) \Rightarrow (q \Rightarrow r))$, prove r.

1.	p	Premise
2.	$p \Rightarrow q$	Premise
3.	$(p \Rightarrow q) \Rightarrow (q \Rightarrow r)$	Premise
4.	q	IE: 2, 1
5.	$q \Rightarrow r$	IE: 3, 2
6.	r	IE: 5, 4

$$\frac{\begin{array}{l}\phi \Rightarrow \psi \\ \phi\end{array}}{\psi}\ \text{IE} \qquad \frac{\psi}{\phi \Rightarrow \psi}\ \text{IC} \qquad \frac{(\phi \Rightarrow (\psi \Rightarrow \chi))}{(\phi \Rightarrow \psi) \Rightarrow (\phi \Rightarrow \chi)}\ \text{ID} \qquad \frac{\neg\psi \Rightarrow \neg\phi}{\phi \Rightarrow \psi}\ \text{IR}$$

Example 2

Given $(p \Rightarrow q)$ and $(q \Rightarrow r)$, prove $(p \Rightarrow r)$.

1.	$p \Rightarrow q$	Premise
2.	$q \Rightarrow r$	Premise

$$\frac{\phi \Rightarrow \psi \quad \phi}{\psi}\ \text{IE} \qquad \frac{\psi}{\phi \Rightarrow \psi}\ \text{IC} \qquad \frac{(\phi \Rightarrow (\psi \Rightarrow \chi))}{(\phi \Rightarrow \psi) \Rightarrow (\phi \Rightarrow \chi)}\ \text{ID} \qquad \frac{\neg\psi \Rightarrow \neg\phi}{\phi \Rightarrow \psi}\ \text{IR}$$

Example 2

Given $(p \Rightarrow q)$ and $(q \Rightarrow r)$, prove $(p \Rightarrow r)$.

1.	$p \Rightarrow q$	Premise
2.	$q \Rightarrow r$	Premise
3.	$p \Rightarrow (q \Rightarrow r)$	IC: 2

$$\frac{\phi \Rightarrow \psi \quad \phi}{\psi}\ \text{IE} \qquad \frac{\psi}{\phi \Rightarrow \psi}\ \text{IC} \qquad \frac{(\phi \Rightarrow (\psi \Rightarrow \chi))}{(\phi \Rightarrow \psi) \Rightarrow (\phi \Rightarrow \chi)}\ \text{ID} \qquad \frac{\neg\psi \Rightarrow \neg\phi}{\phi \Rightarrow \psi}\ \text{IR}$$

Example 2

Given $(p \Rightarrow q)$ and $(q \Rightarrow r)$, prove $(p \Rightarrow r)$.

1.	$p \Rightarrow q$	Premise
2.	$q \Rightarrow r$	Premise
3.	$p \Rightarrow (q \Rightarrow r)$	IC: 2
4.	$(p \Rightarrow q) \Rightarrow (p \Rightarrow r)$	ID: 3

$$\frac{\phi \Rightarrow \psi \qquad \phi}{\psi}\ \text{IE} \qquad \frac{\psi}{\phi \Rightarrow \psi}\ \text{IC} \qquad \frac{(\phi \Rightarrow (\psi \Rightarrow \chi))}{(\phi \Rightarrow \psi) \Rightarrow (\phi \Rightarrow \chi)}\ \text{ID} \qquad \frac{\neg\psi \Rightarrow \neg\phi}{\phi \Rightarrow \psi}\ \text{IR}$$

Example 2

Given $(p \Rightarrow q)$ and $(q \Rightarrow r)$, prove $(p \Rightarrow r)$.

1.	$p \Rightarrow q$	Premise
2.	$q \Rightarrow r$	Premise
3.	$p \Rightarrow (q \Rightarrow r)$	IC: 2
4.	$(p \Rightarrow q) \Rightarrow (p \Rightarrow r)$	ID: 3
5.	$(p \Rightarrow r)$	IE: 4, 1

$$\frac{\begin{array}{l}\phi \Rightarrow \psi \\ \phi\end{array}}{\psi}\ \text{IE} \qquad \frac{\psi}{\phi \Rightarrow \psi}\ \text{IC} \qquad \frac{(\phi \Rightarrow (\psi \Rightarrow \chi))}{(\phi \Rightarrow \psi) \Rightarrow (\phi \Rightarrow \chi)}\ \text{ID} \qquad \frac{\neg\psi \Rightarrow \neg\phi}{\phi \Rightarrow \psi}\ \text{IR}$$

Example 3

Given p and $\neg p$, prove q.

1.	p	Premise
2.	$\neg p$	Premise

$$\frac{\phi \Rightarrow \psi \quad \phi}{\psi}\ \text{IE} \qquad \frac{\psi}{\phi \Rightarrow \psi}\ \text{IC} \qquad \frac{(\phi \Rightarrow (\psi \Rightarrow \chi))}{(\phi \Rightarrow \psi) \Rightarrow (\phi \Rightarrow \chi)}\ \text{ID} \qquad \frac{\neg\psi \Rightarrow \neg\phi}{\phi \Rightarrow \psi}\ \text{IR}$$

Example 3

Given p and $\neg p$, prove q.

1.	p	Premise
2.	$\neg p$	Premise
3.	$\neg q \Rightarrow \neg p$	IC: 2

$$\frac{\phi \Rightarrow \psi \quad \phi}{\psi}\ \text{IE} \qquad \frac{\psi}{\phi \Rightarrow \psi}\ \text{IC} \qquad \frac{(\phi \Rightarrow (\psi \Rightarrow \chi))}{(\phi \Rightarrow \psi) \Rightarrow (\phi \Rightarrow \chi)}\ \text{ID} \qquad \frac{\neg \psi \Rightarrow \neg \phi}{\phi \Rightarrow \psi}\ \text{IR}$$

Example 3

Given p and $\neg p$, prove q.

1.	p	Premise
2.	$\neg p$	Premise
3.	$\neg q \Rightarrow \neg p$	IC: 2
4.	$p \Rightarrow q$	IR: 3

$$\frac{\begin{array}{l}\phi \Rightarrow \psi \\ \phi\end{array}}{\psi}$$
IE

$$\frac{\psi}{\phi \Rightarrow \psi}$$
IC

$$\frac{(\phi \Rightarrow (\psi \Rightarrow \chi))}{(\phi \Rightarrow \psi) \Rightarrow (\phi \Rightarrow \chi)}$$
ID

$$\frac{\neg \psi \Rightarrow \neg \phi}{\phi \Rightarrow \psi}$$
IR

Example 3

Given p and $\neg p$, prove q.

1.	p	Premise
2.	$\neg p$	Premise
3.	$\neg q \Rightarrow \neg p$	IC: 2
4.	$p \Rightarrow q$	IR: 3
5.	q	IE: 4, 1

$$\frac{\phi \Rightarrow \psi \quad \phi}{\psi} \text{ IE} \qquad \frac{\psi}{\phi \Rightarrow \psi} \text{ IC} \qquad \frac{(\phi \Rightarrow (\psi \Rightarrow \chi))}{(\phi \Rightarrow \psi) \Rightarrow (\phi \Rightarrow \chi)} \text{ ID} \qquad \frac{\neg \psi \Rightarrow \neg \phi}{\phi \Rightarrow \psi} \text{ IR}$$

Example 4

Given $\neg\neg p$, prove p.

1. $\neg\neg p$ Premise

$$\frac{\begin{array}{l}\phi \Rightarrow \psi \\ \phi\end{array}}{\psi}\ \text{IE} \qquad \frac{\psi}{\phi \Rightarrow \psi}\ \text{IC} \qquad \frac{(\phi \Rightarrow (\psi \Rightarrow \chi))}{(\phi \Rightarrow \psi) \Rightarrow (\phi \Rightarrow \chi)}\ \text{ID} \qquad \frac{\neg\psi \Rightarrow \neg\phi}{\phi \Rightarrow \psi}\ \text{IR}$$

Example 4

Given $\neg\neg p$, prove p.

1.	$\neg\neg p$	Premise
2.	$\neg\neg\neg\neg p \Rightarrow \neg\neg p$	IC: 1

$$\frac{\begin{array}{l}\phi \Rightarrow \psi \\ \phi\end{array}}{\psi}\ \text{IE} \qquad \frac{\psi}{\phi \Rightarrow \psi}\ \text{IC} \qquad \frac{(\phi \Rightarrow (\psi \Rightarrow \chi))}{(\phi \Rightarrow \psi) \Rightarrow (\phi \Rightarrow \chi)}\ \text{ID} \qquad \frac{\neg\psi \Rightarrow \neg\phi}{\phi \Rightarrow \psi}\ \text{IR}$$

Example 4

Given $\neg\neg p$, prove p.

1.	$\neg\neg p$	Premise
2.	$\neg\neg\neg\neg p \Rightarrow \neg\neg p$	IC: 1
3.	$\neg p \Rightarrow \neg\neg\neg p$	IR: 2
4.	$\neg\neg p \Rightarrow p$	IR: 3

$$\frac{\phi \Rightarrow \psi \quad \phi}{\psi}\ \text{IE} \qquad \frac{\psi}{\phi \Rightarrow \psi}\ \text{IC} \qquad \frac{(\phi \Rightarrow (\psi \Rightarrow \chi))}{(\phi \Rightarrow \psi) \Rightarrow (\phi \Rightarrow \chi)}\ \text{ID} \qquad \frac{\neg\psi \Rightarrow \neg\phi}{\phi \Rightarrow \psi}\ \text{IR}$$

Example 4

Given $\neg\neg p$, prove p.

1.	$\neg\neg p$	Premise
2.	$\neg\neg\neg\neg p \Rightarrow \neg\neg p$	IC: 1
3.	$\neg p \Rightarrow \neg\neg\neg p$	IR: 2
4.	$\neg\neg p \Rightarrow p$	IR: 3
5	p	IE: 4, 1

$$\frac{\phi \Rightarrow \psi \quad \phi}{\psi}\ \text{IE} \qquad \frac{\psi}{\phi \Rightarrow \psi}\ \text{IC} \qquad \frac{(\phi \Rightarrow (\psi \Rightarrow \chi))}{(\phi \Rightarrow \psi) \Rightarrow (\phi \Rightarrow \chi)}\ \text{ID} \qquad \frac{\neg\psi \Rightarrow \neg\phi}{\phi \Rightarrow \psi}\ \text{IR}$$

Proof Systems

Proof Systems

A proof system is a finite set of (usually valid) axiom schemata and (usually sound) rules of inference.

Examples:

- Hilbert / Lukasiewicz
- Frege
- Mendelson
- Meredith

Hilbert System

Rule of Inference (IE):

$$\frac{\phi \Rightarrow \psi \qquad \phi}{\psi}$$

Theoretically interesting.

Practically useless.

Pedagogical value.

Axiom Schemata (due to Lukasiewicz):

IC: $\varphi \Rightarrow (\psi \Rightarrow \varphi)$

ID: $(\varphi \Rightarrow (\psi \Rightarrow \chi)) \Rightarrow ((\varphi \Rightarrow \psi) \Rightarrow (\varphi \Rightarrow \chi))$

IR: $(\neg\psi \Rightarrow \neg\varphi) \Rightarrow (\varphi \Rightarrow \psi)$

NB: Any PL sentence can be converted algorithmically into a logically equivalent sentence involving just $\neg$ and $\Rightarrow$. (Substitution Theorem and validities.)

Example 1 - Transitivity

Given $(p \Rightarrow q)$ and $(q \Rightarrow r)$, prove $(p \Rightarrow r)$.

1.	$p \Rightarrow q$	Premise
2.	$q \Rightarrow r$	Premise
3.	$(q \Rightarrow r) \Rightarrow (p \Rightarrow (q \Rightarrow r))$	IC
4.	$(p \Rightarrow (q \Rightarrow r))$	IE: 2, 3
5.	$(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$	ID
6.	$(p \Rightarrow q) \Rightarrow (p \Rightarrow r)$	IE: 4, 5
7.	$p \Rightarrow r$	IE: 1, 6

Example 2 - Reflexivity

Prove $(p \Rightarrow p)$.

1.	$p \Rightarrow (p \Rightarrow p)$	IC
2.	$p \Rightarrow ((p \Rightarrow p) \Rightarrow p)$	IC
3.	$p \Rightarrow ((p \Rightarrow p) \Rightarrow p) \Rightarrow ((p \Rightarrow (p \Rightarrow p)) \Rightarrow (p \Rightarrow p))$	ID
4.	$(p \Rightarrow (p \Rightarrow p)) \Rightarrow (p \Rightarrow p)$	IE: $2, 3$
5.	$(p \Rightarrow p)$	IE: $1, 4$

Example 3 - Inconsistency

Given p and $\neg p$, prove q.

1.	p	Premise
2.	$\neg p$	Premise
3	$\neg p \Rightarrow (\neg q \Rightarrow \neg p)$	IC
4	$\neg q \Rightarrow \neg p$	IE: $3, 2$
5.	$(\neg q \Rightarrow \neg p) \Rightarrow (p \Rightarrow q)$	IR
6.	$p \Rightarrow q$	IE: $5, 4$
7.	q	IE: $6, 1$

Meredith System

Rule of Inference:

$$\frac{\begin{array}{l}\phi \Rightarrow \psi \\ \phi\end{array}}{\psi}$$

Axiom Schema:

$$(((((\phi \Rightarrow \psi) \Rightarrow (\neg\chi \Rightarrow \neg\theta)) \Rightarrow \chi) \Rightarrow \boldsymbol{\tau}) \Rightarrow$$
$$((\boldsymbol{\tau} \Rightarrow \phi) \Rightarrow (\theta \Rightarrow \phi)))$$

Meredith System

Unfortunately:

Proofs hard to generate

Proofs hard to understand

Theoretically interesting.

Practically useless.

No pedagogical value.

Rules and Axiom Schemata

Axiom Schemata as 0-ary Rules of Inference

$$\varphi \Rightarrow \varphi \qquad\qquad \frac{}{\varphi \Rightarrow \varphi}$$

Rules of Inference as Axiom Schemata

$$\frac{\neg\psi \Rightarrow \neg\varphi}{\varphi \Rightarrow \psi} \qquad\qquad (\varphi \Rightarrow \psi) \Rightarrow (\neg\psi \Rightarrow \neg\varphi)$$

NB: We must keep at least one rule of inference. By convention, we retain Implication Elimination.

Axiom Schemata and Rules

Fact: If a sentence is valid, then it is true under all interpretations. Consequently, there should be a proof without making any assumptions at all.

Fact: $(p \Rightarrow p)$ is a valid sentence.

Problem: Prove $(p \Rightarrow p)$ without any premises.

NB: We must have some schemata or 0-ary rules of inference.

Soundness and Completeness

Provability

If there exists a proof of a sentence ϕ from a set Δ of premises using the rules of inference in R, we say that ϕ is *provable* from Δ using R.

We usually write this as $\Delta \vdash_R \phi$, using the provability operator $\vdash$ (which is sometimes called *single turnstile*). (If the set of rules is clear from context, we usually drop the subscript, writing just $\Delta \vdash \phi$.)

$$\{p \Rightarrow q, q \Rightarrow r\} \vdash (p \Rightarrow r)$$

Logical Entailment and Provability

A set of premises Δ *logically entails* a conclusion φ ($\Delta \models \varphi$) if and only if every interpretation that satisfies Δ also satisfies φ.

If there exists a proof of a sentence ϕ from a set Δ of premises using the rules of inference in R, we say that ϕ is *provable* from Δ using R (written $\Delta \vdash_R \phi$).

Soundness and Completeness

A proof system is *sound* if and only if every provable conclusion is logically entailed.

$$\text{If } \Delta \vdash \phi \text{, then } \Delta \vDash \phi.$$

A proof system is *complete* if and only if every logically entailed conclusion is provable.

$$\text{If } \Delta \vDash \phi \text{, then } \Delta \vdash \phi.$$

Hilbert System

Theorem: The Hilbert System is sound and complete for Propositional Logic.

$$\Delta \vdash_{\text{Hilbert}} \phi \text{ if and only if } \Delta \vDash \phi.$$

Translation: *The truth table method and the proof method succeed in exactly the same cases.*

NB: The Meredith System is also sound and complete.

$$\Delta \vdash_{\text{Meredith}} \phi \text{ if and only if } \Delta \vDash \phi.$$

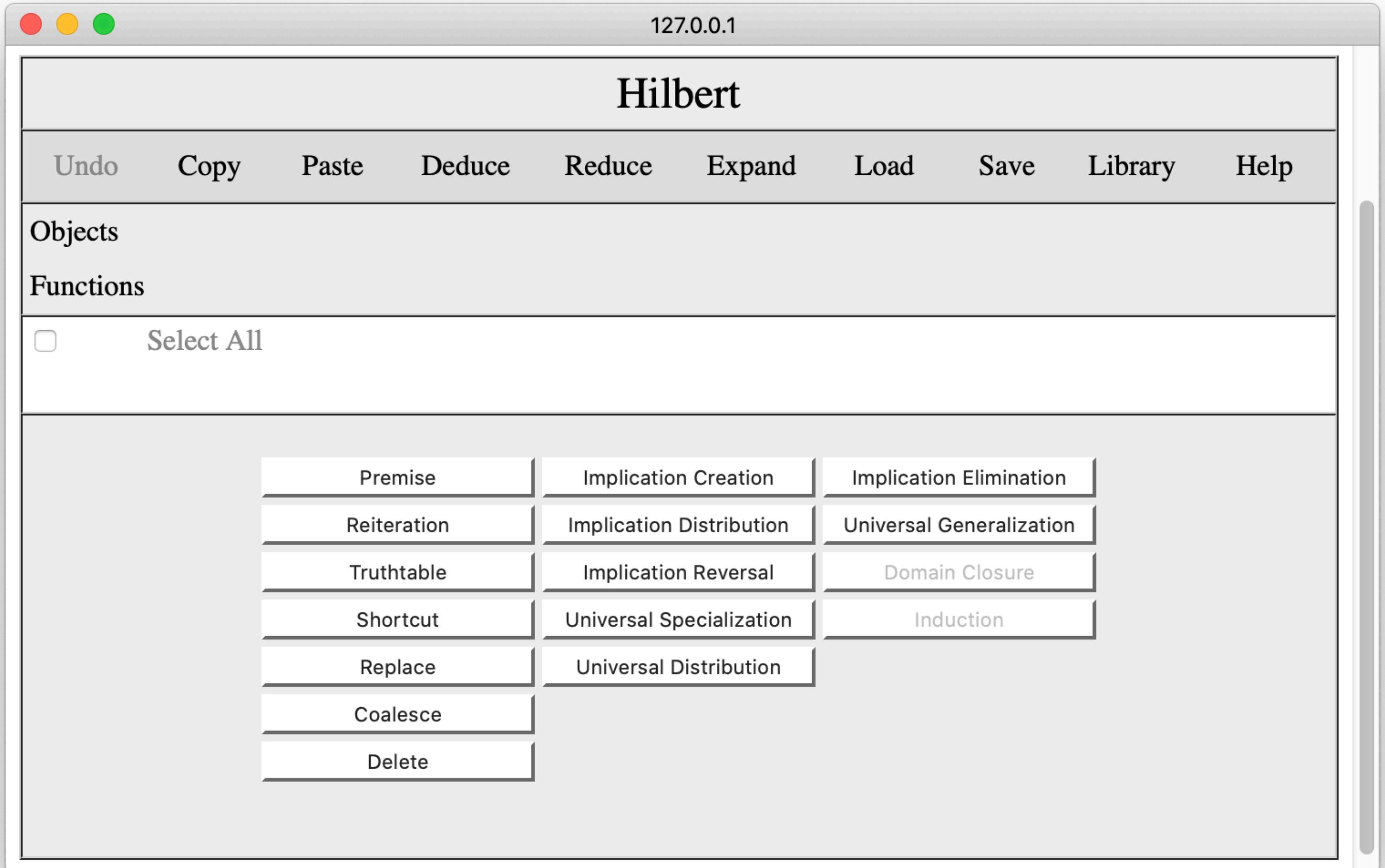
Significance

Proofs are often much smaller than the corresponding truth tables.

Finding a proof often takes fewer steps than the truth table method. (However, in the worst case, this may take just as many or even more steps.)

Hilbert Online System

http://logica.stanford.edu/documentation/hilbert.html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Implication Elimination
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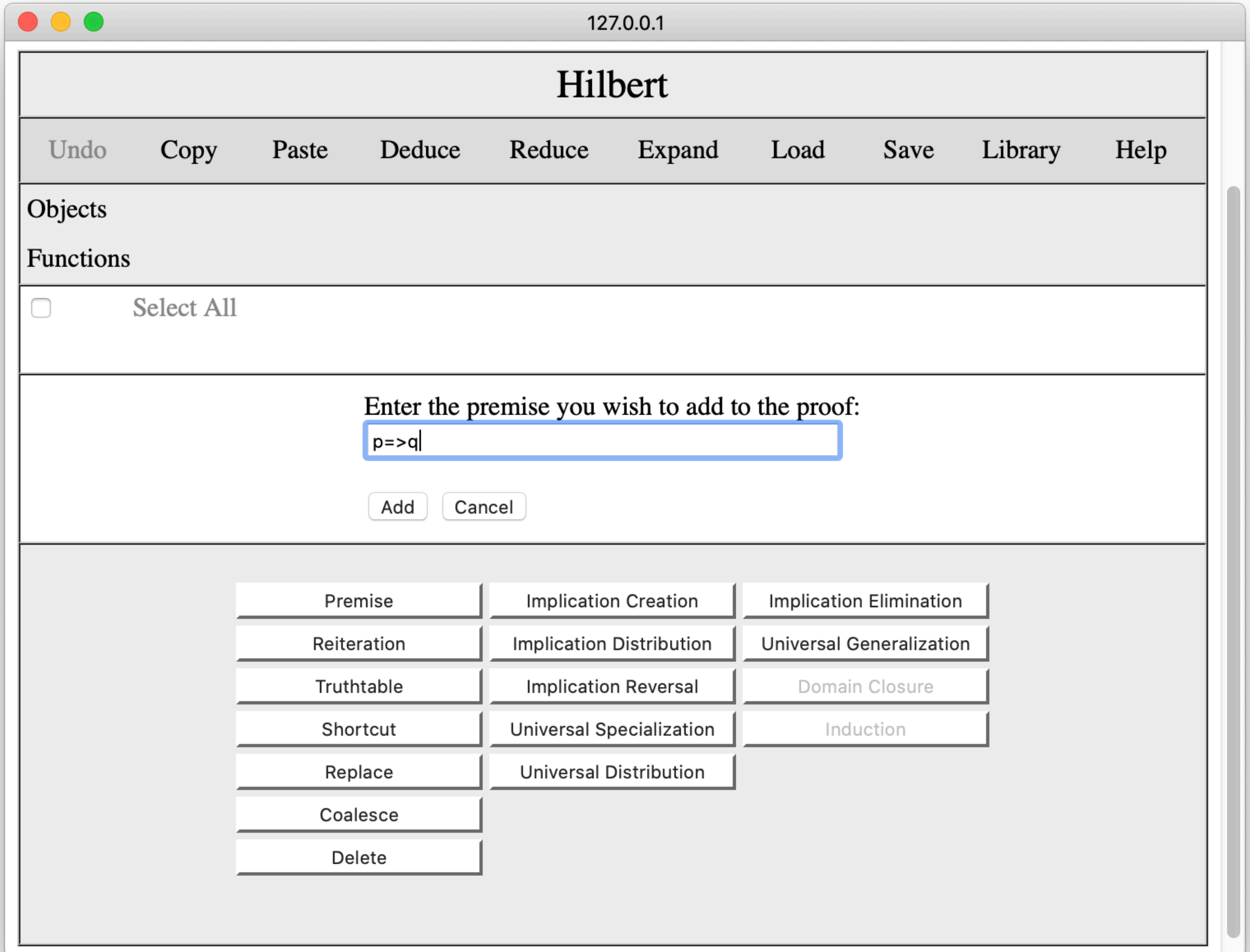
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Enter the premise you wish to add to the proof:
p=>q
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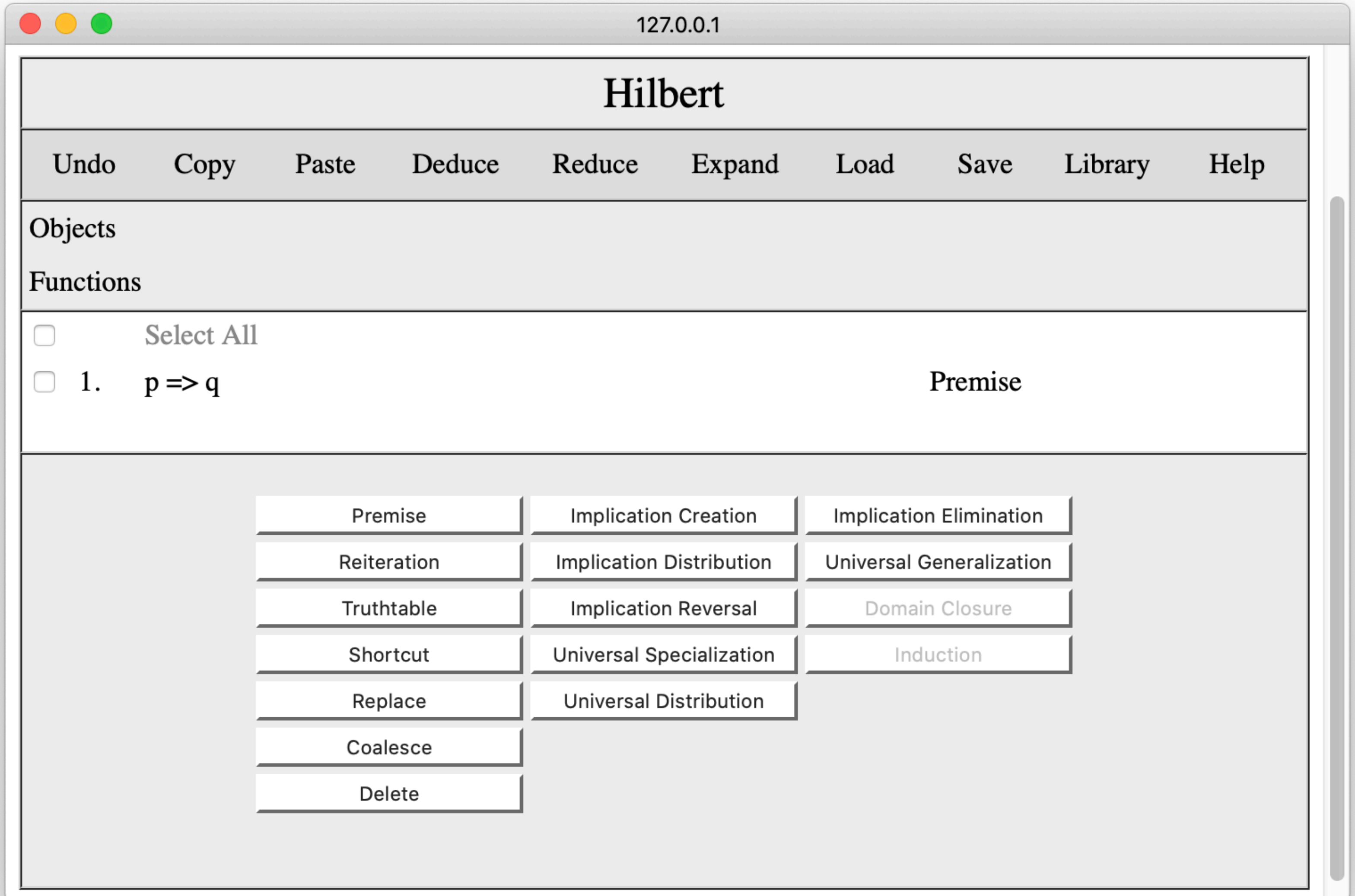
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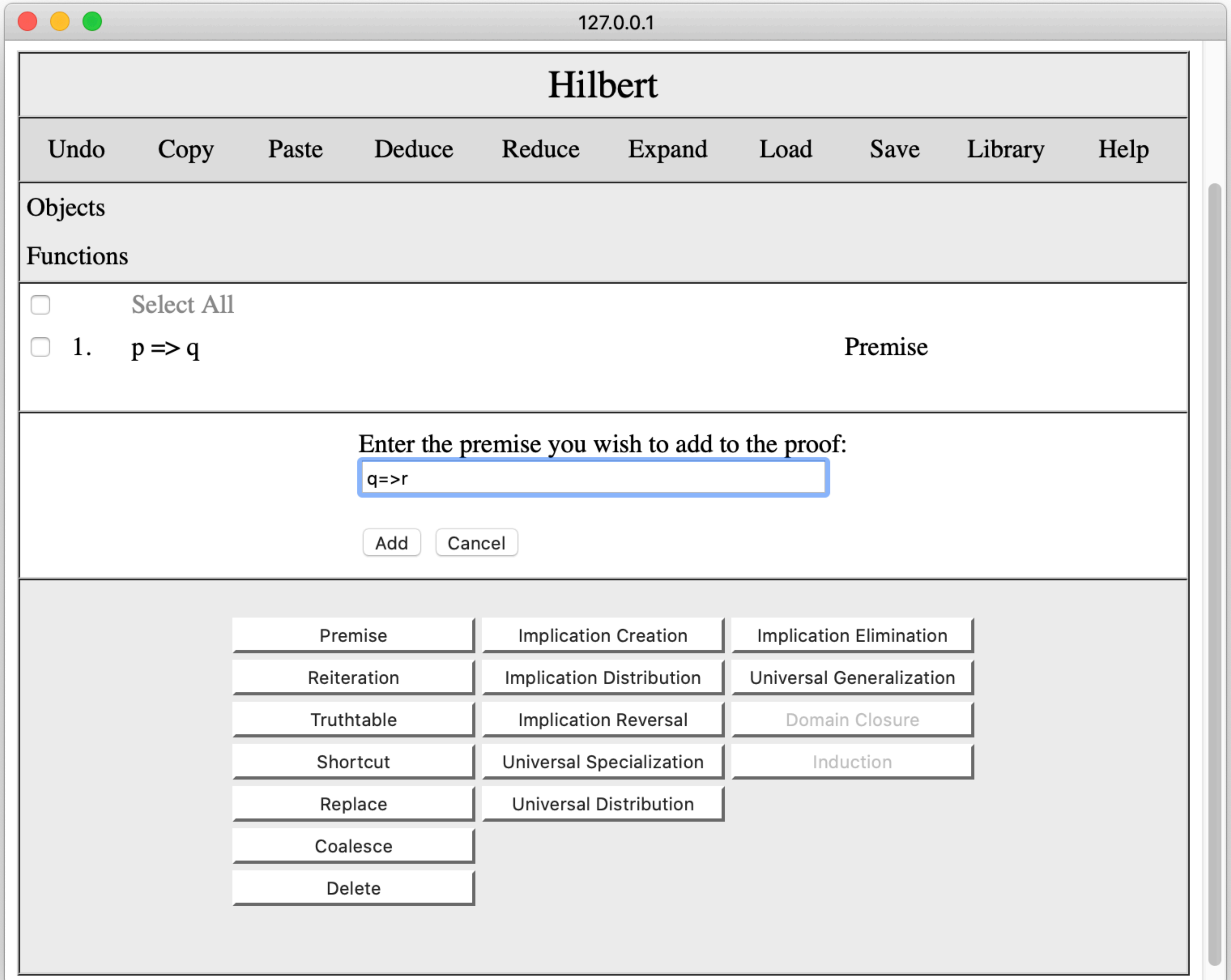
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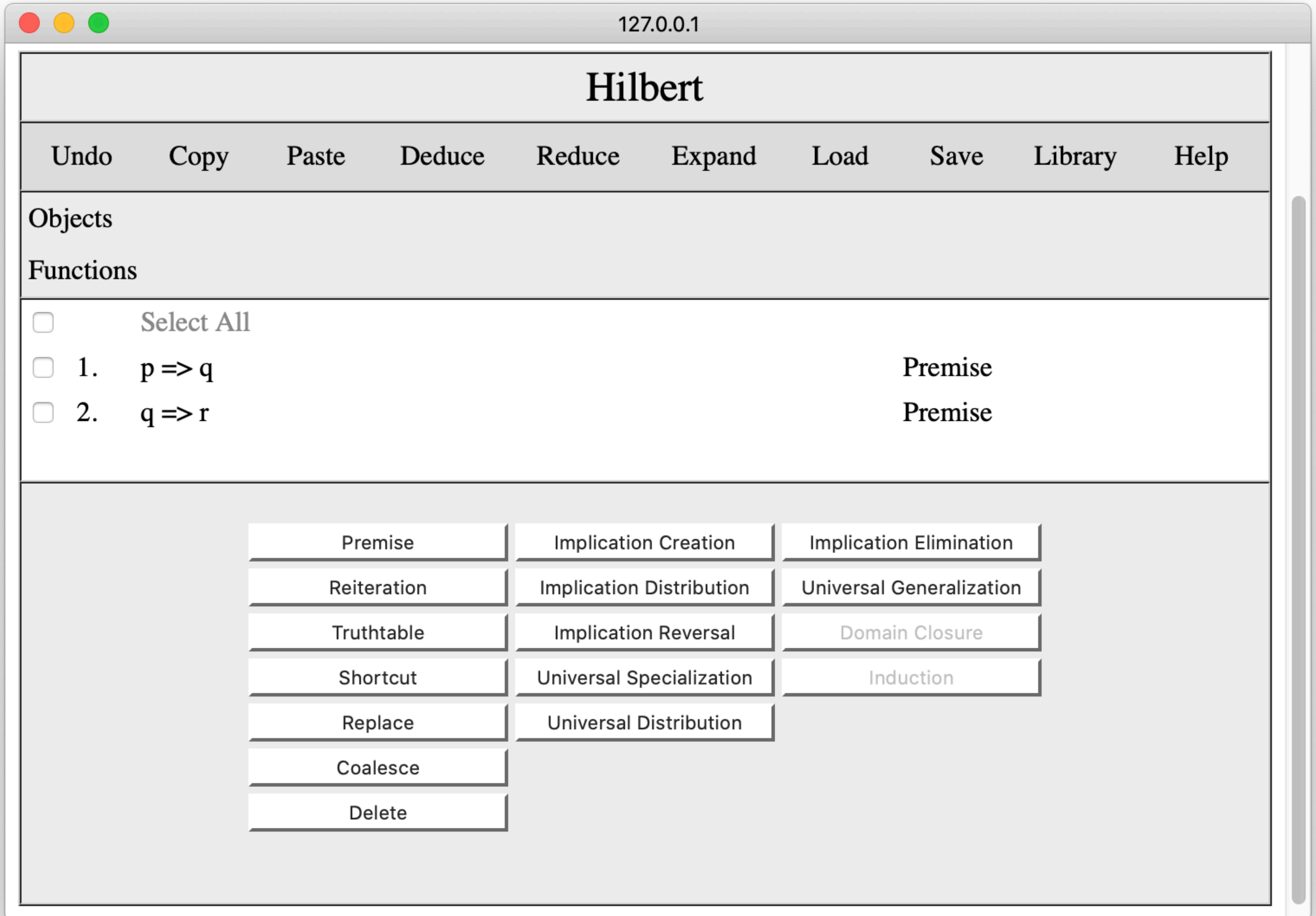
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Premise
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Implication Introduction:
ϕ => (ψ => ϕ)
Enter values for the meta-variables in the schema:
ϕ: q=>r
ψ: p
Add
Cancel
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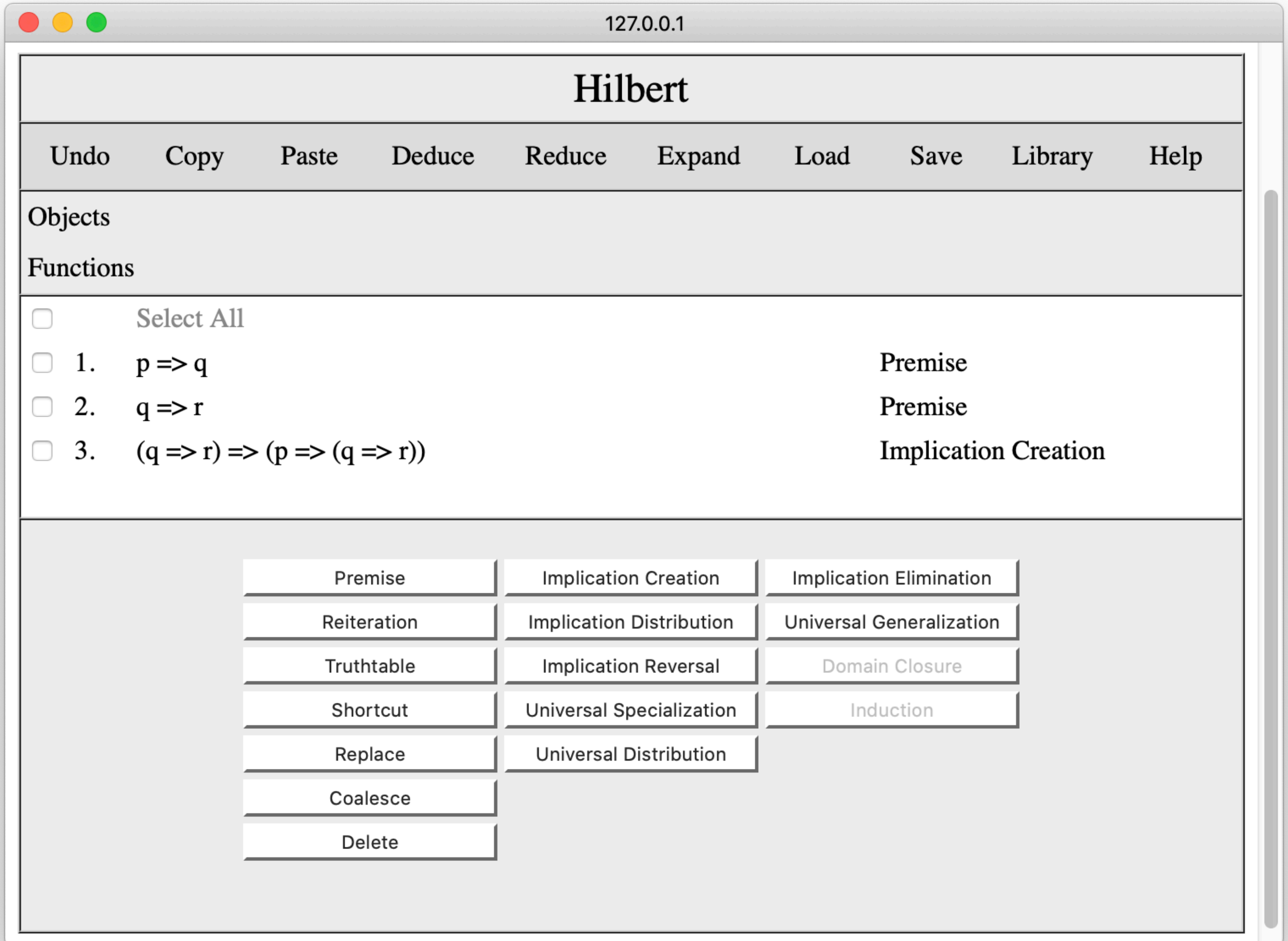
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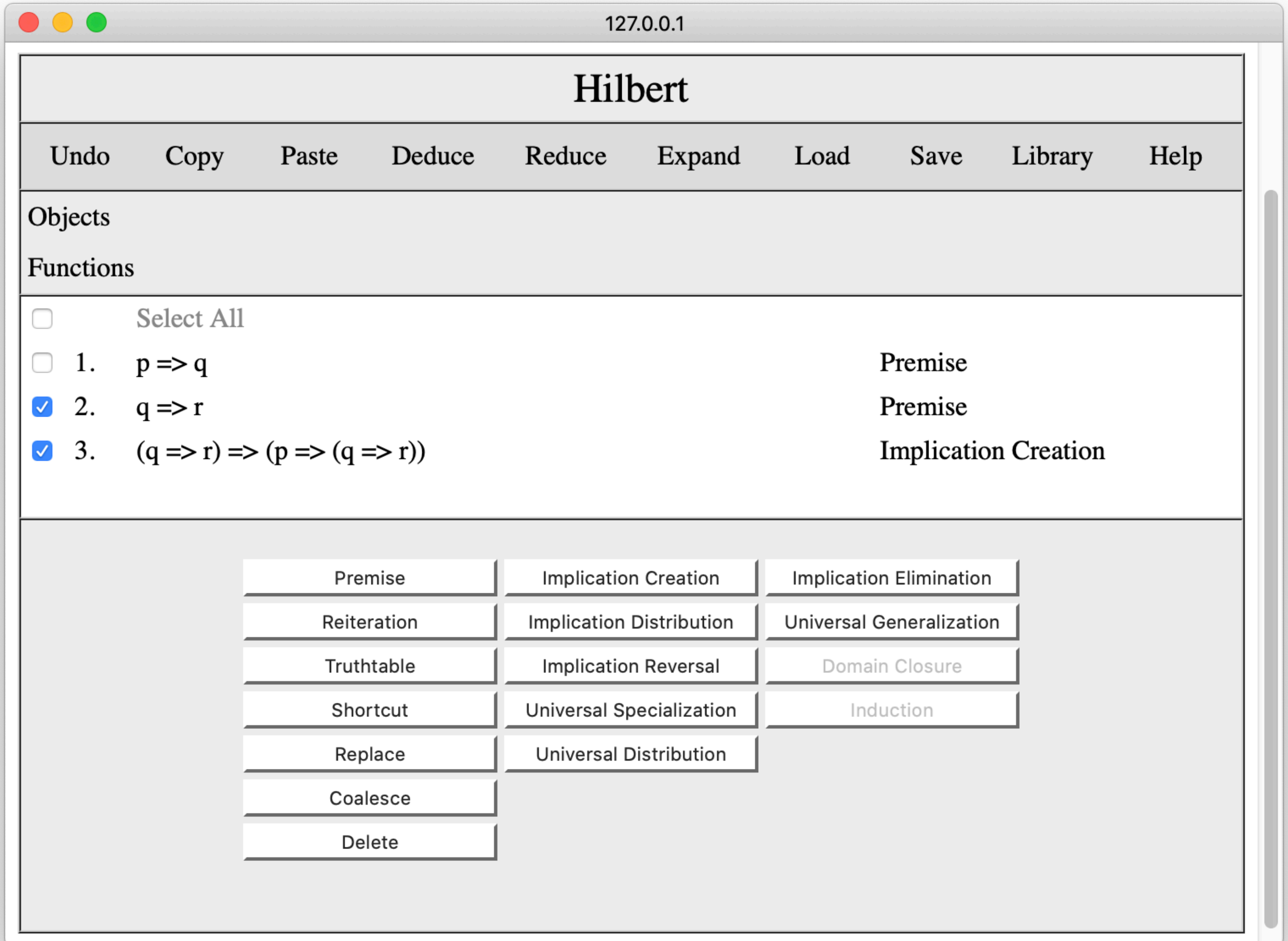
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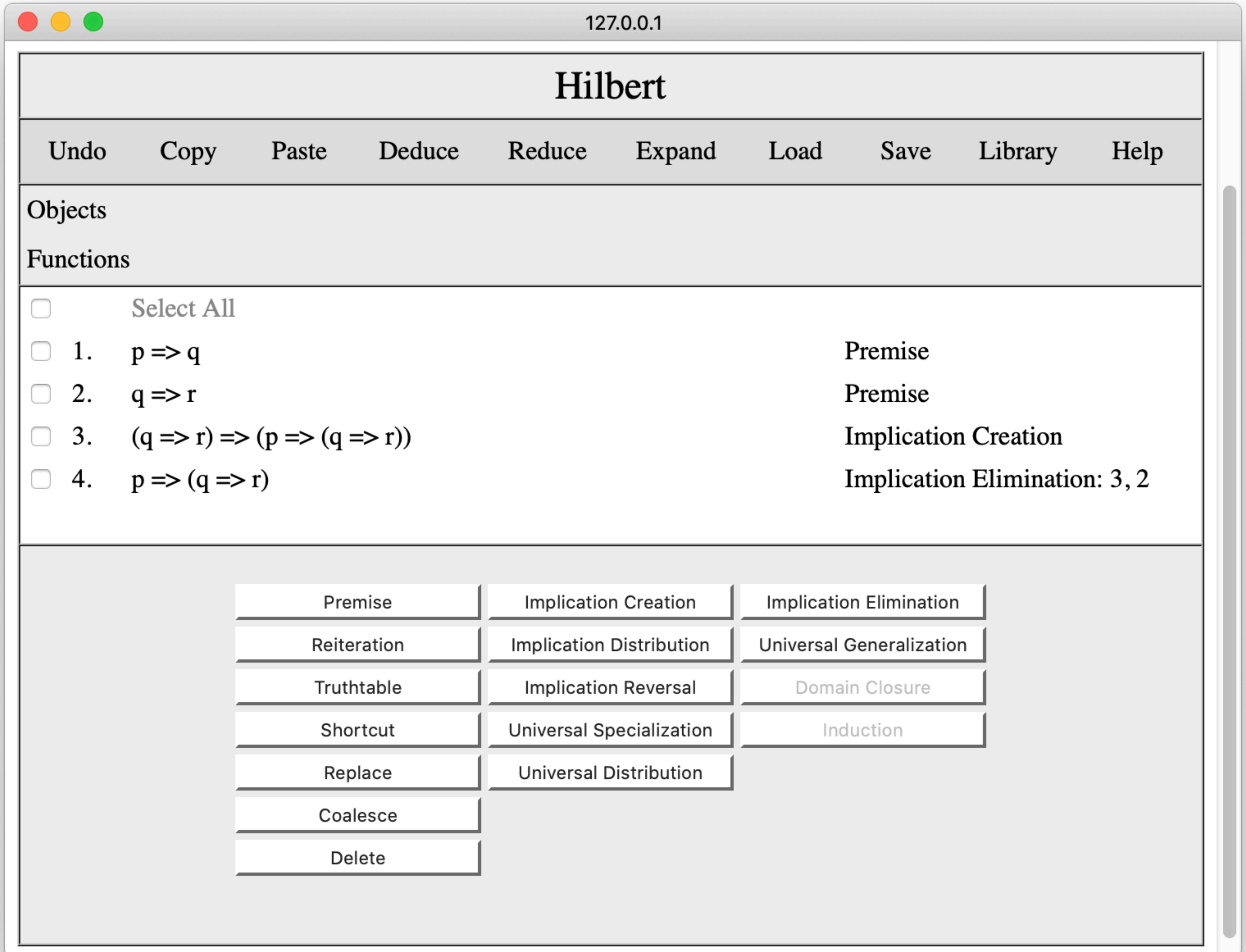
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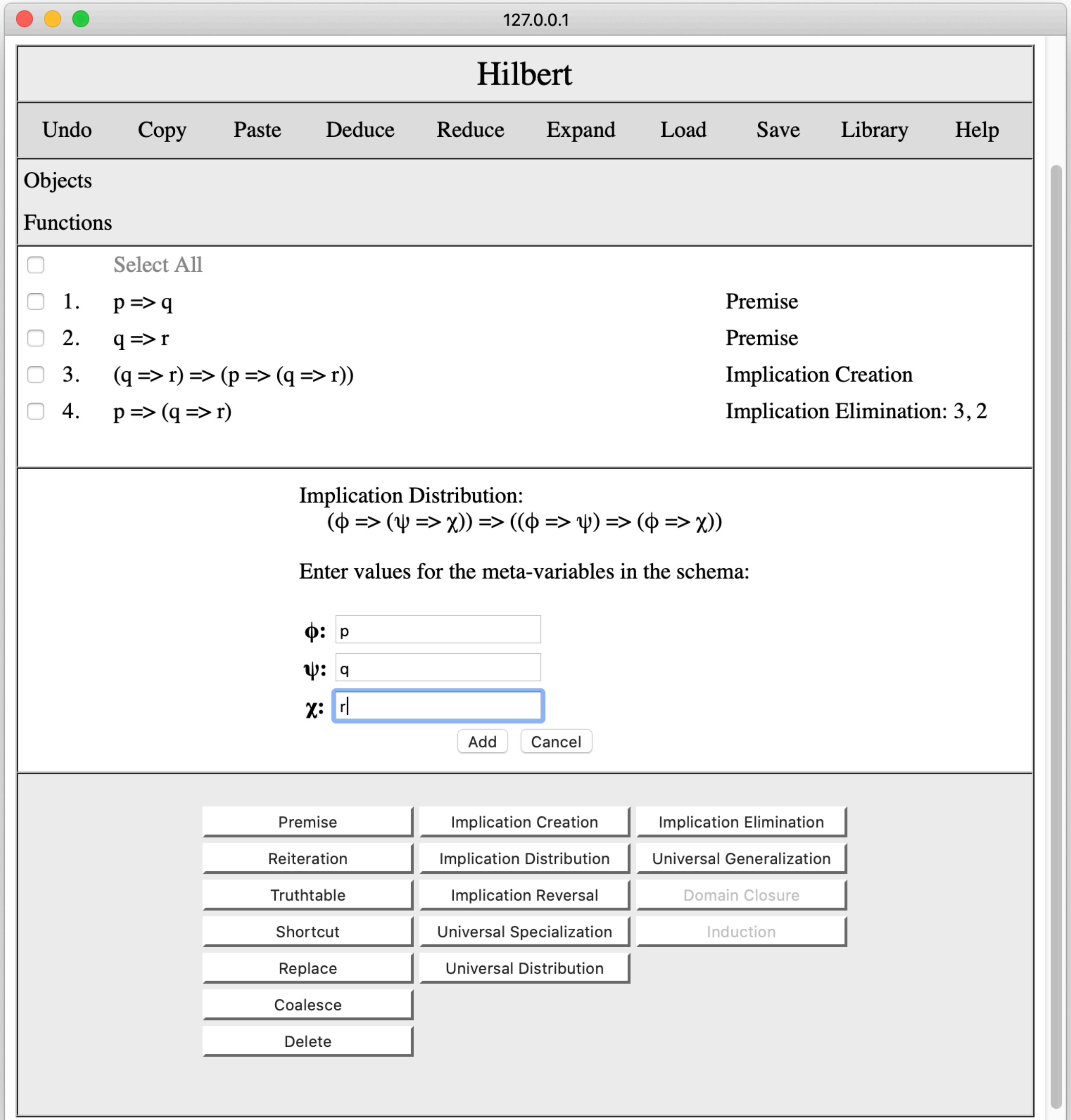
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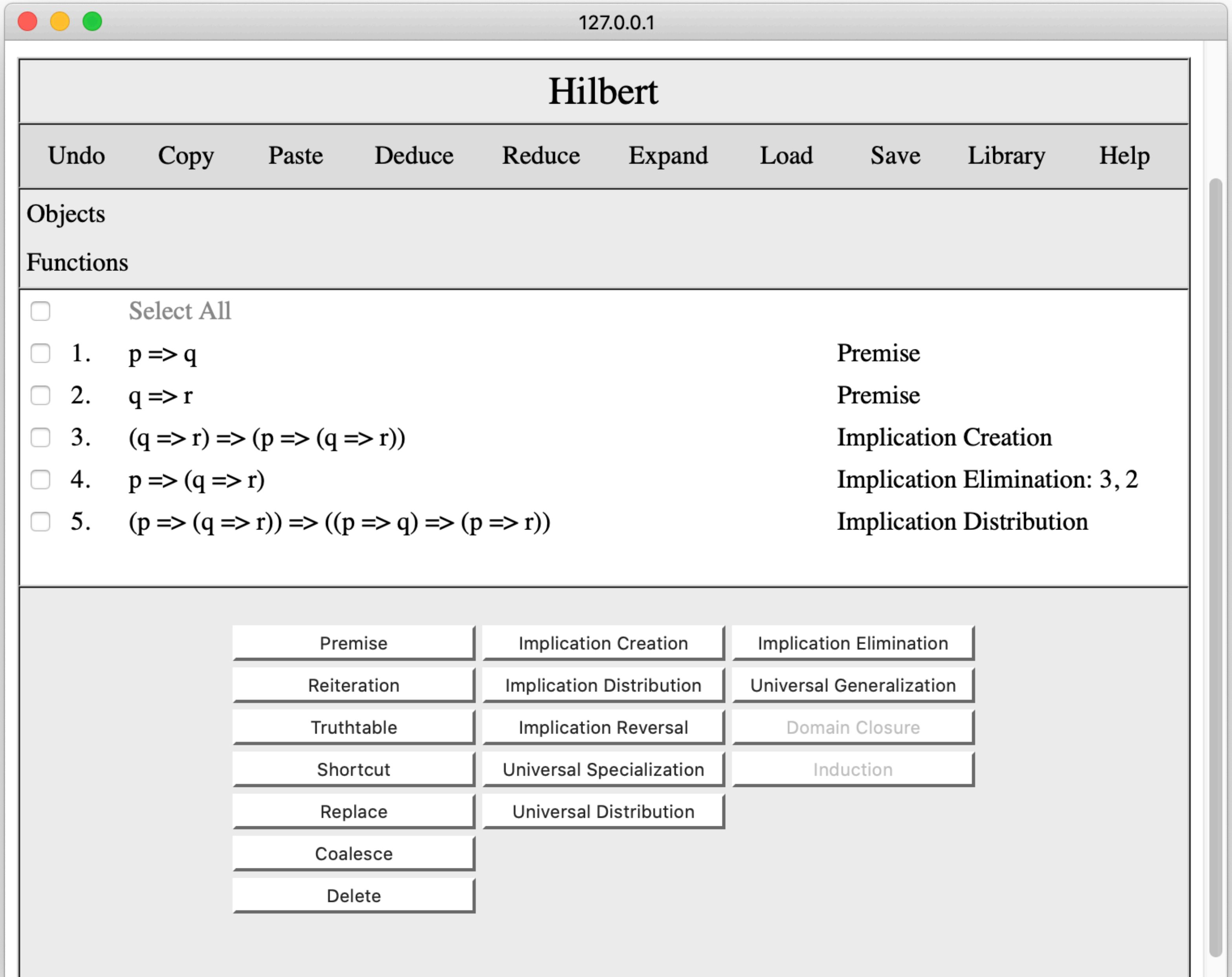
Undo Copy Paste Deduce Reduce Expand Load Save Library Help
Objects
Functions
Select All
1. p => q Premise
2. q => r Premise
3. (q => r) => (p => (q => r)) Implication Creation
4. p => (q => r) Implication Elimination: 3, 2
Implication Distribution:
(φ => (ψ => χ)) => ((φ => ψ) => (φ => χ))
Enter values for the meta-variables in the schema:
φ: p
ψ: q
χ: r
Add
Cancel
Premise
Implication Creation
Implication Elimination
Reiteration
Implication Distribution
Universal Generalization
Truthtable
Implication Reversal
Domain Closure
Shortcut
Universal Specialization
Induction
Replace
Universal Distribution
Coalesce
Delete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1. p => q Premise
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4. p => (q => r) Implication Elimination: 3, 2
5. (p => (q => r)) => ((p => q) => (p => r)) Implication Distribution
Premise
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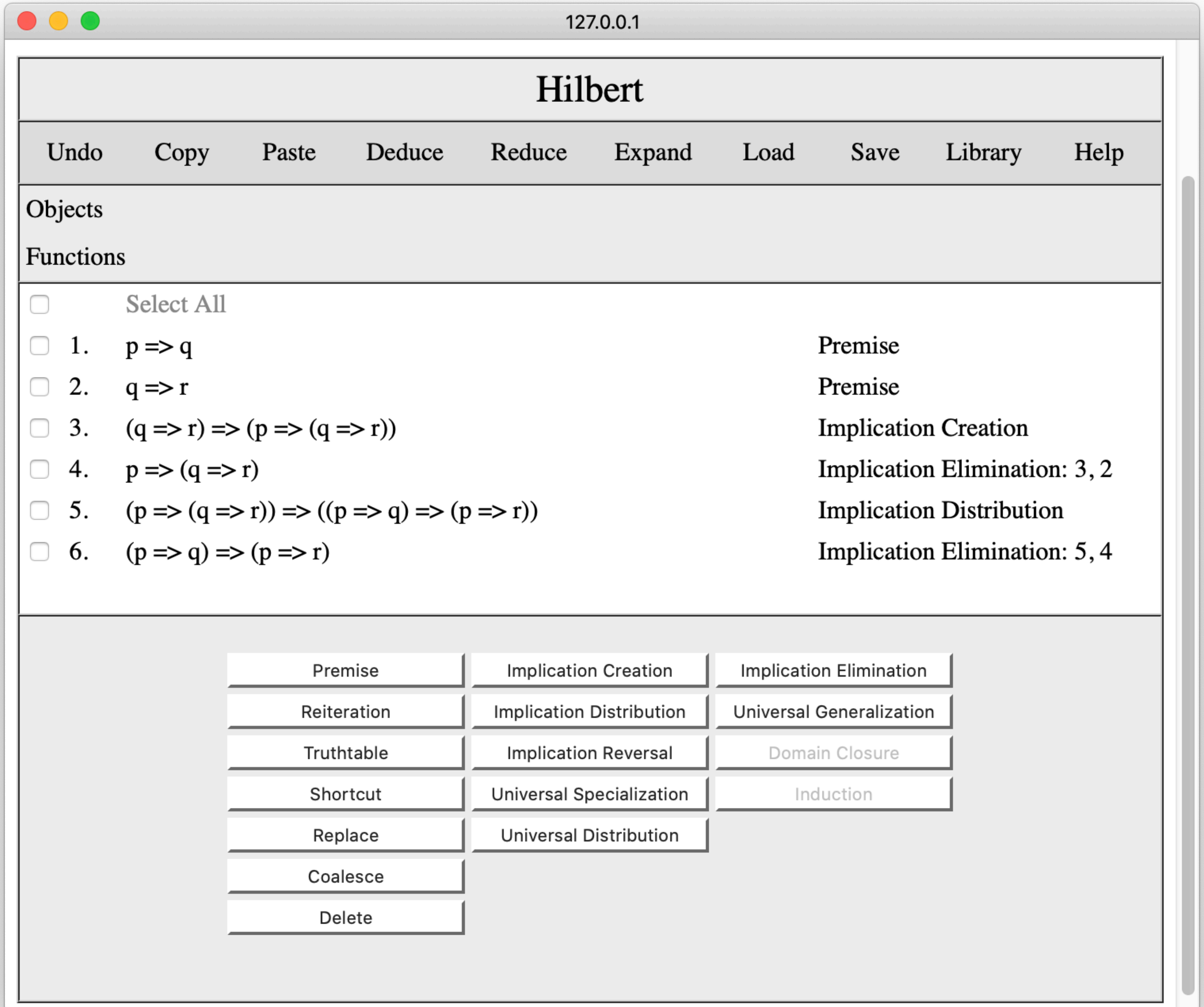
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Objects
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2. q => r Premise
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5. (p => (q => r)) => ((p => q) => (p => r)) Implication Distribution
6. (p => q) => (p => r) Implication Elimination: 5, 4
Premise
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Implication Elimination
Reiteration
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Truthtable
Implication Reversal
Domain Closure
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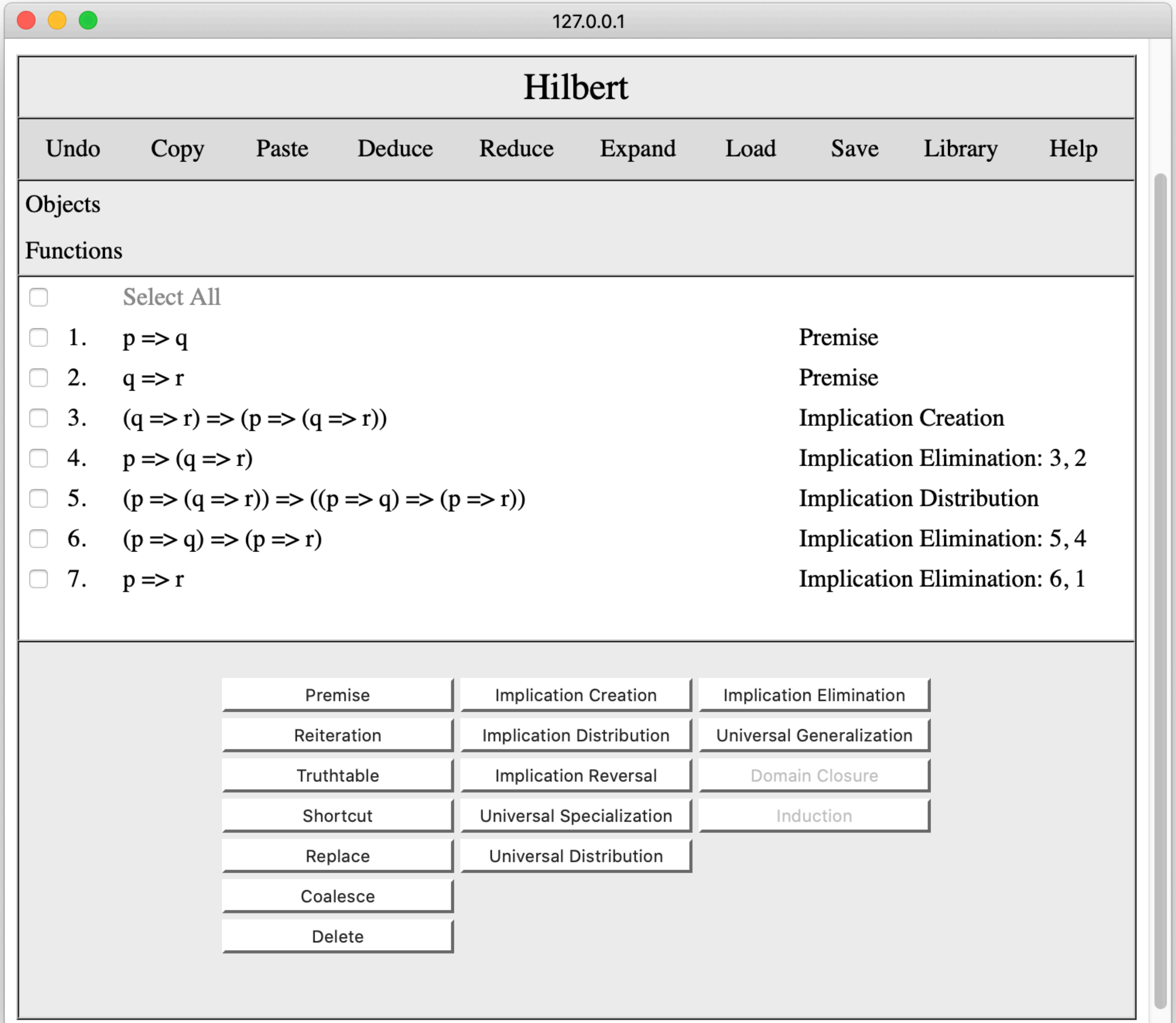
Universal Distribution
Coalesce
Delete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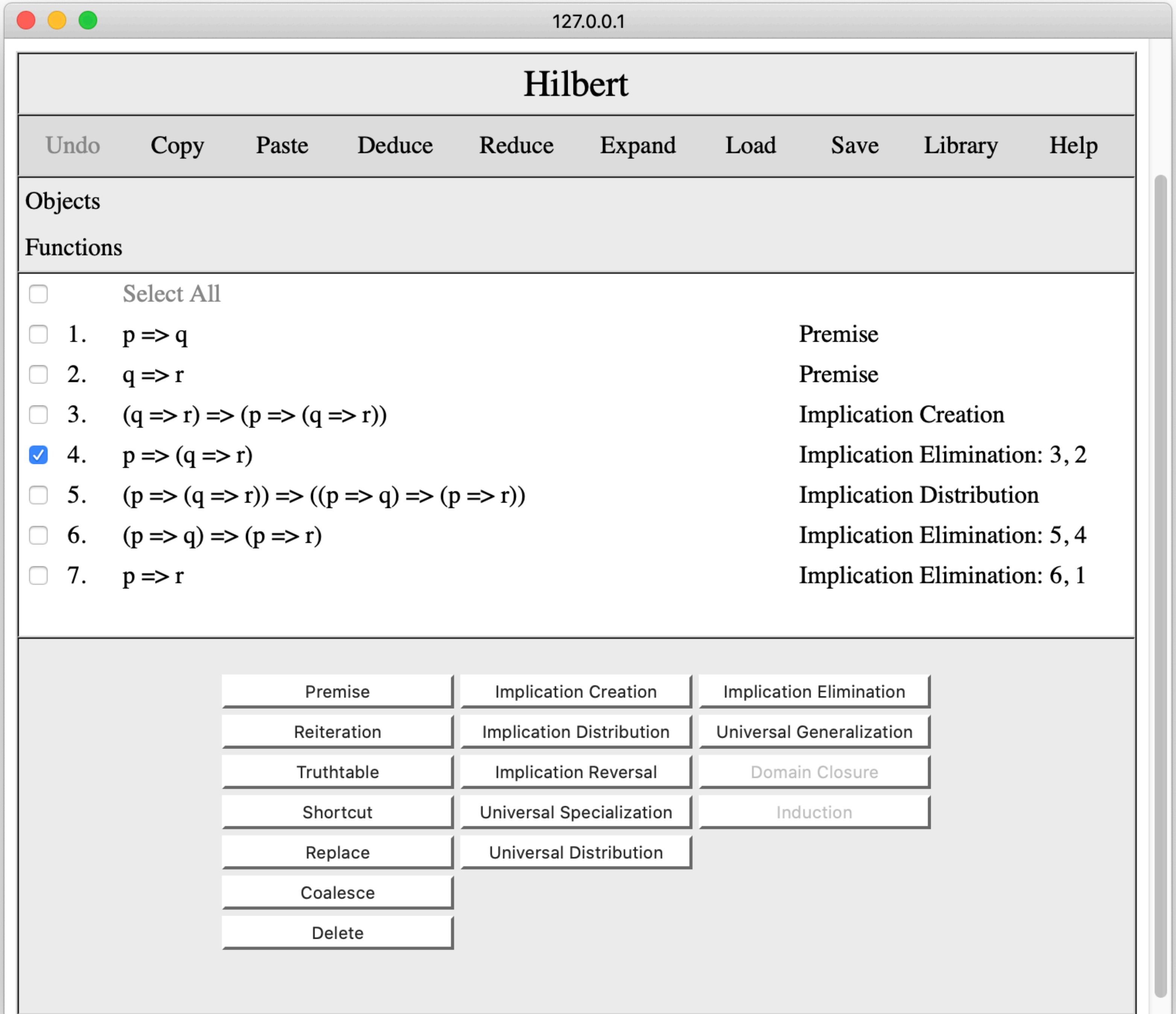
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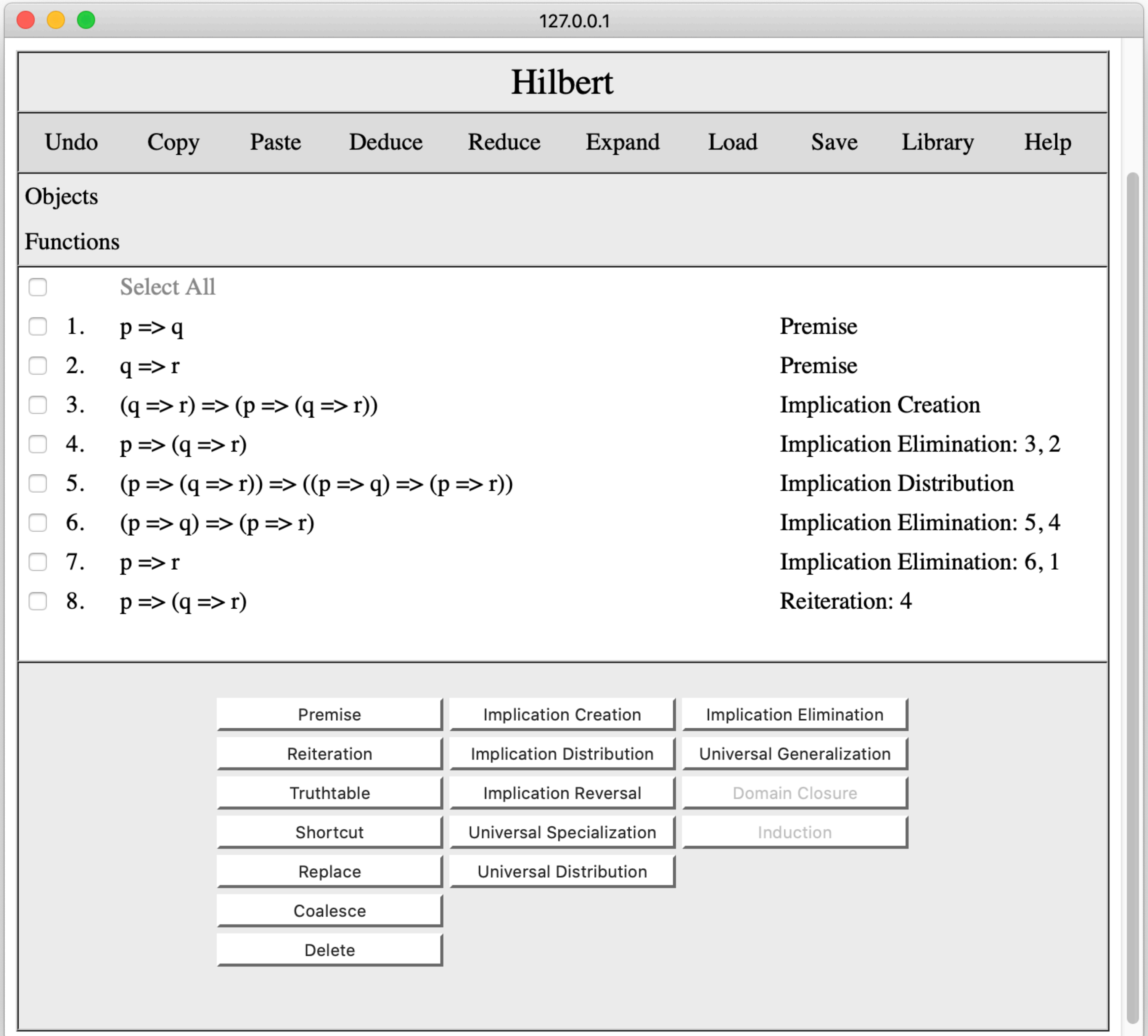
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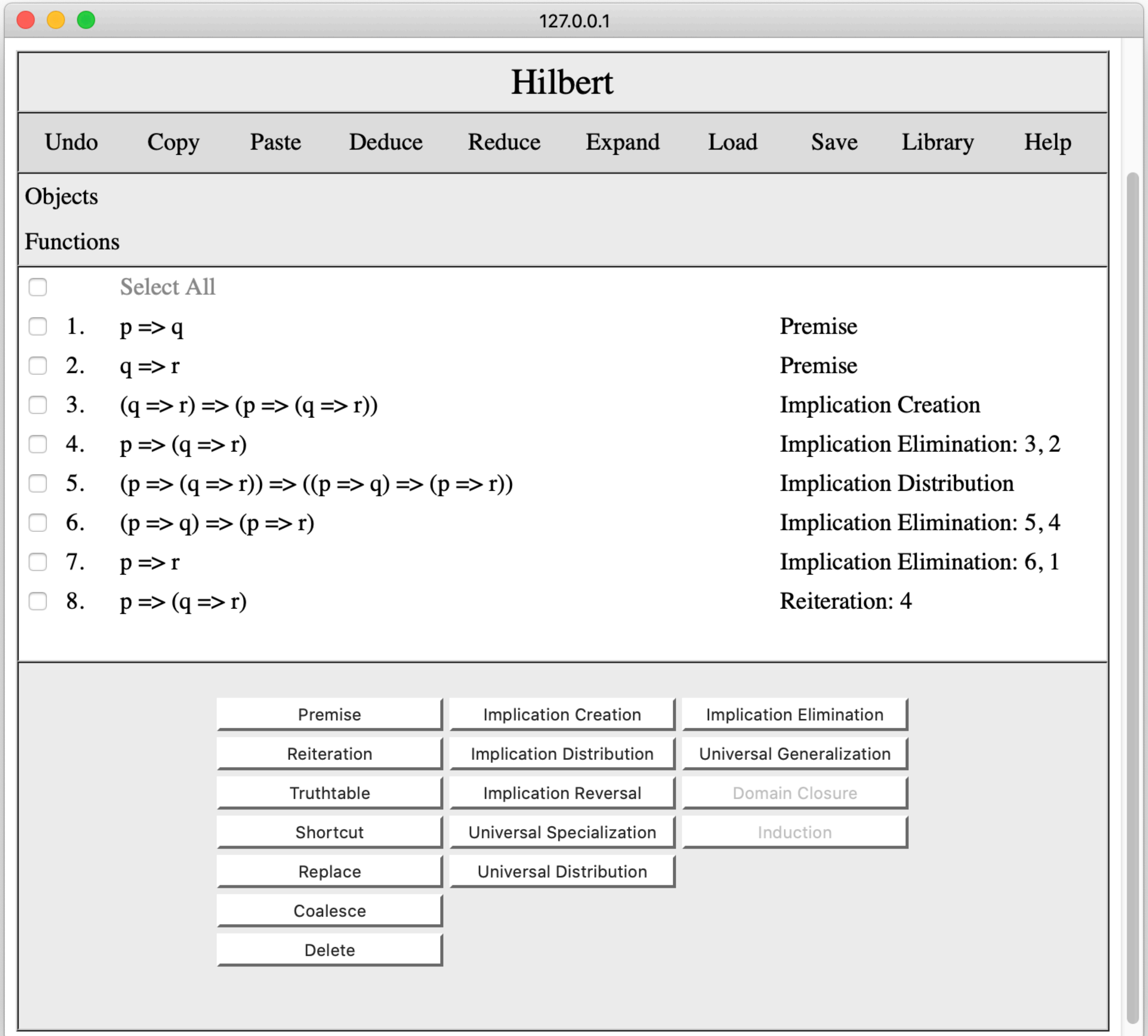
6. (p => q) => (p => r) Implication Elimination: 5, 4
7. p => r Implication Elimination: 6, 1
8. p => (q => r) Reiteration: 4
Premise
Implication Creation
Implication Elimination
Reiteration
Implication Distribution
Universal Generalization
Truthtable
Implication Reversal
Domain Closure
Shortcut
Universal Specialization
Induction
Replace
Universal Distribution
Coalesce
Delete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6. (p => q) => (p => r) Implication Elimination: 5, 4
7. p => r Implication Elimination: 6, 1
8. p => (q => r) Reiteration: 4
Premise
Implication Creation
Implication Elimination
Reiteration
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Universal Generalization
Truthtable
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Domain Closure
Shortcut
Universal Specialization
Induction
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Universal Distribution
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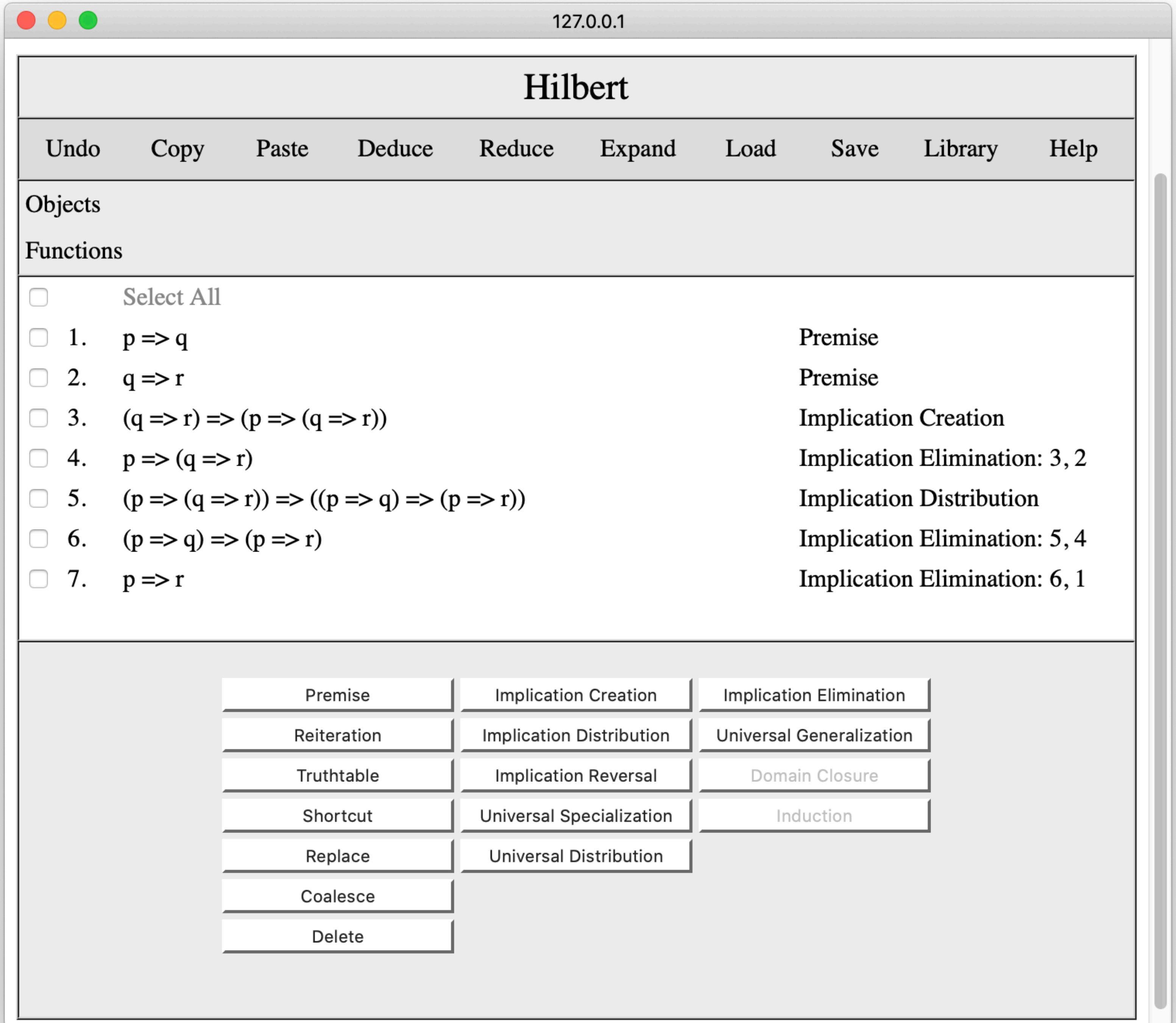
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2. q => r Premise
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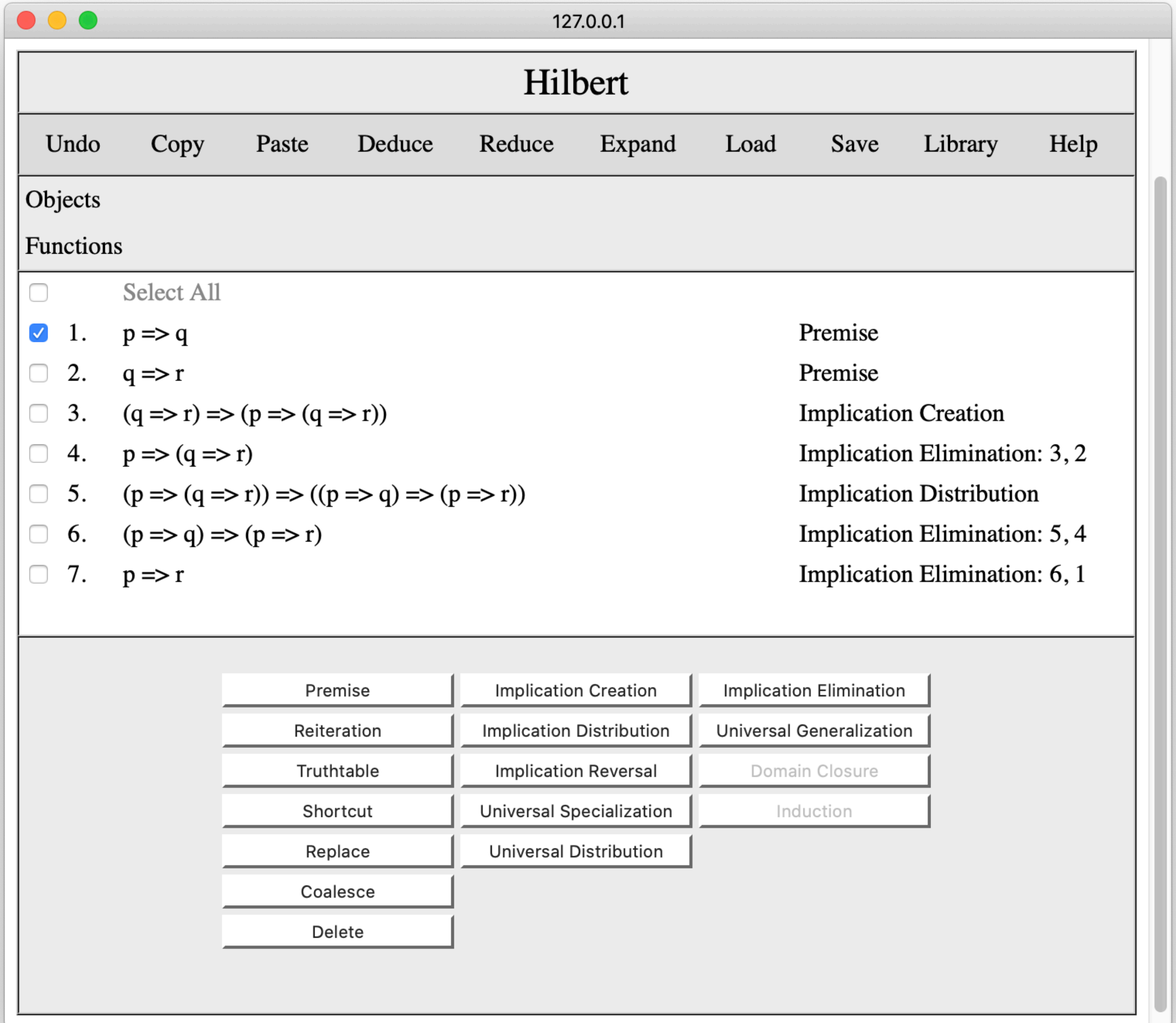
4. p => (q => r) Implication Elimination: 3, 2
5. (p => (q => r)) => ((p => q) => (p => r)) Implication Distribution
6. (p => q) => (p => r) Implication Elimination: 5, 4
7. p => r Implication Elimination: 6, 1
Premise
Implication Creation
Implication Elimination
Reiteration
Implication Distribution
Universal Generalization
Truthtable
Implication Reversal
Domain Closure
Shortcut
Universal Specialization
Induction
Replace
Universal Distribution
Coalesce
Delete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Objects
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Select All
1. p => q Premise
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3. (q => r) => (p => (q => r)) Implication Creation
4. p => (q => r) Implication Elimination: 3, 2
5. (p => (q => r)) => ((p => q) => (p => r)) Implication Distribution
6. (p => q) => (p => r) Implication Elimination: 5, 4
7. p => r Implication Elimination: 6, 1
Premise
Implication Creation
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Universal Generalization
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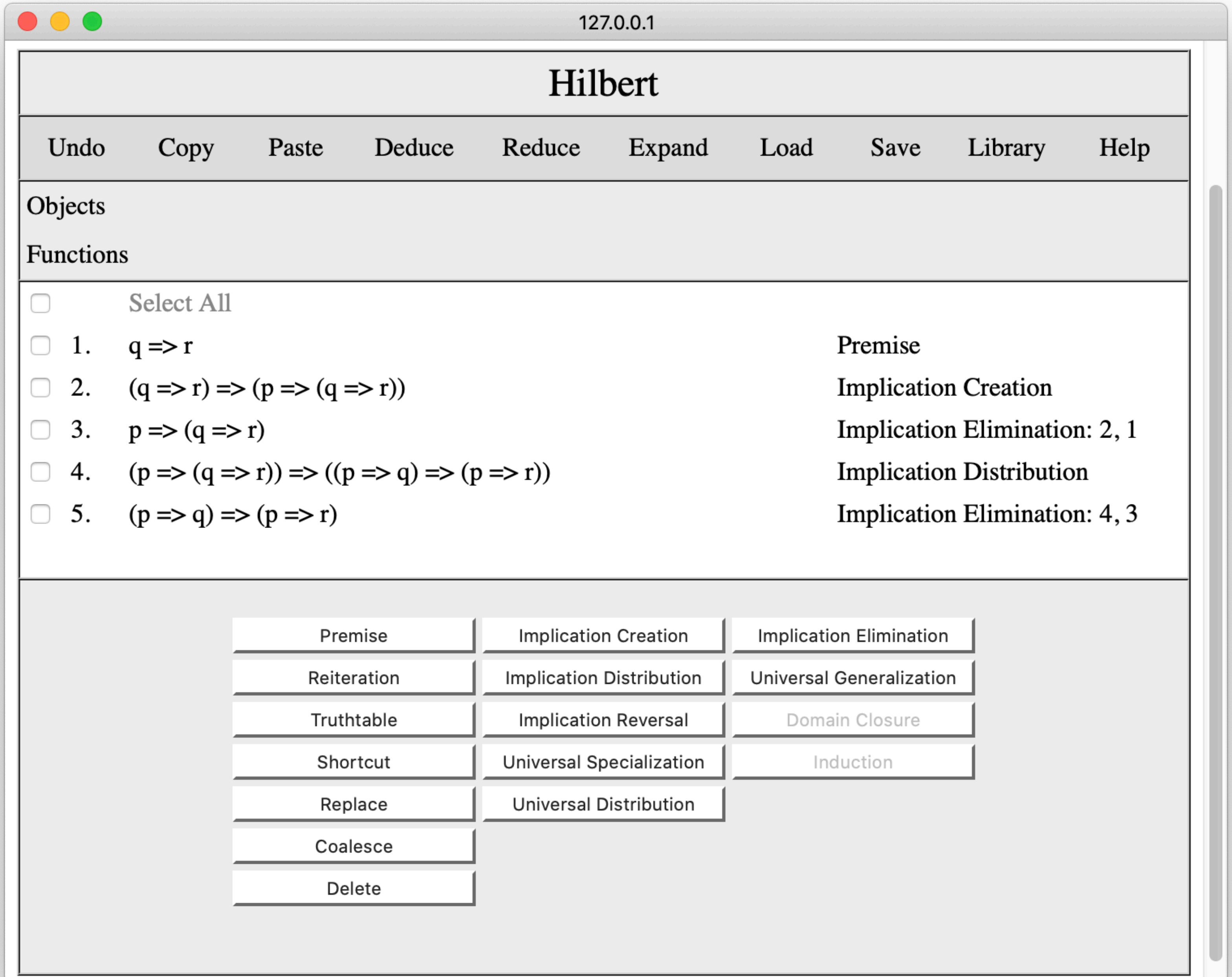
Universal Specialization
Induction
Replace
Universal Distribution
Coalesce
Delete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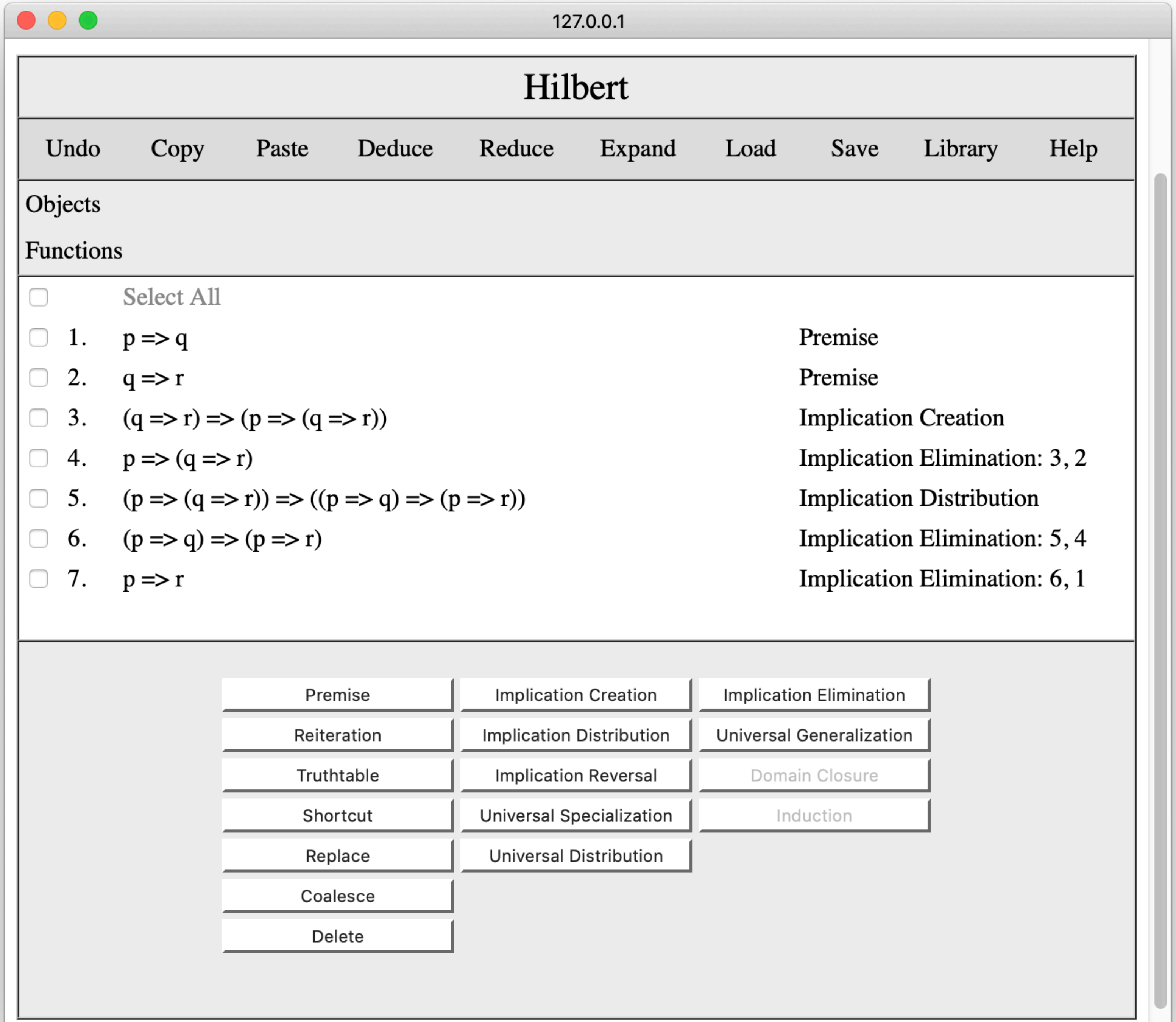
Objects
Functions
Select All
1. q => r Premise
2. (q => r) => (p => (q => r)) Implication Creation
3. p => (q => r) Implication Elimination: 2, 1
4. (p => (q => r)) => ((p => q) => (p => r)) Implication Distribution
5. (p => q) => (p => r) Implication Elimination: 4, 3
Premise
Implication Creation
Implication Elimination
Reiteration
Implication Distribution
Universal Generalization
Truthtable
Implication Reversal
Domain Closure
Shortcut
Universal Specialization
Induction
Replace
Universal Distribution
Coalesce
Delete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Reduce
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Library
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Objects
Functions
Select All
1. p => q
Premise
2. q => r
Premise
3. (q => r) => (p => (q => r))
Implication Creation
4. p => (q => r)
Implication Elimination: 3, 2
5. (p => (q => r)) => ((p => q) => (p => r))
Implication Distribution
6. (p => q) => (p => r)
Implication Elimination: 5, 4
7. p => r
Implication Elimination: 6, 1
Premise
Implication Creation
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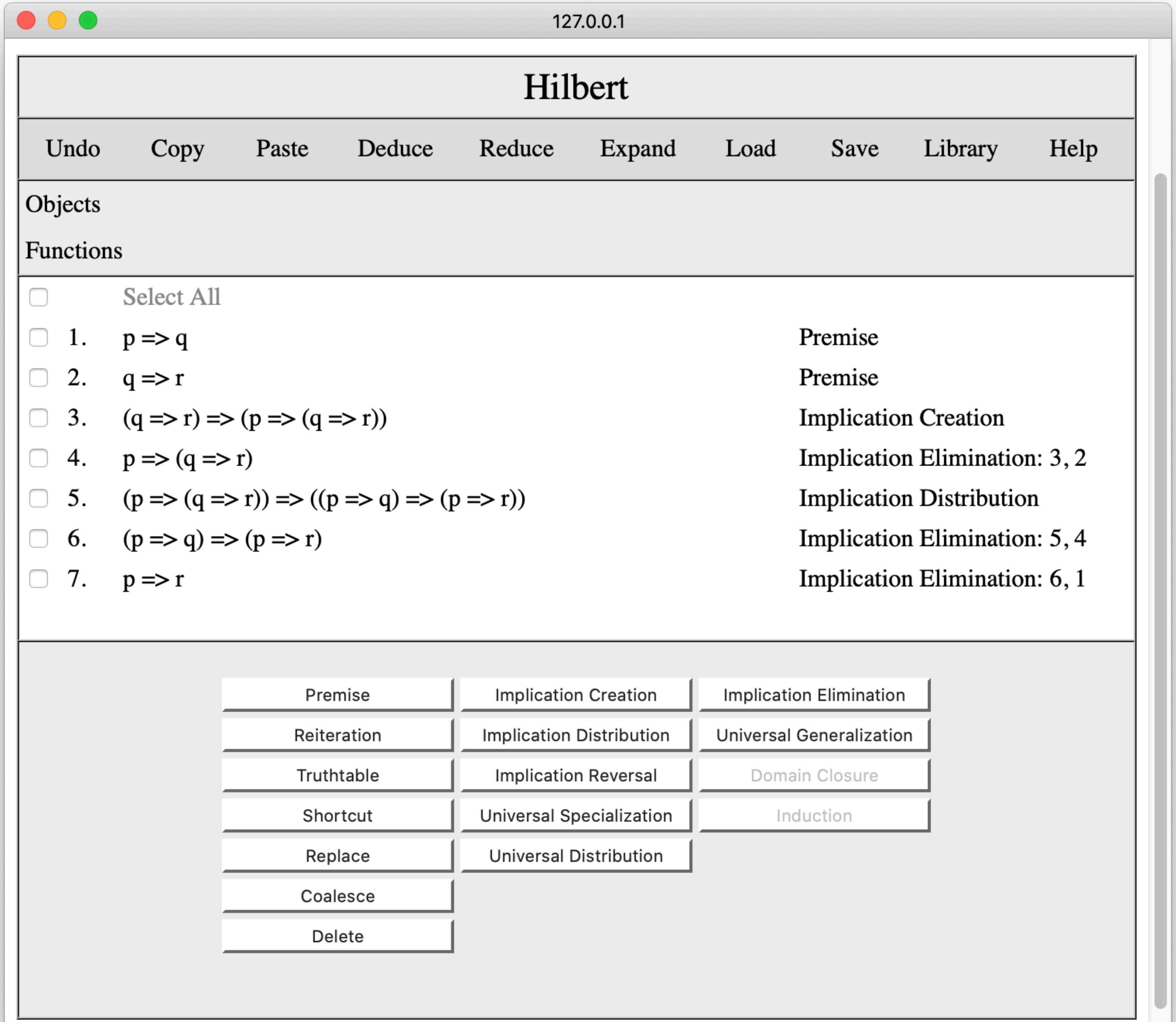
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Truthtable
Implication Reversal
Domain Closure
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Universal Specialization
Induction
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Universal Distribution
Coalesce
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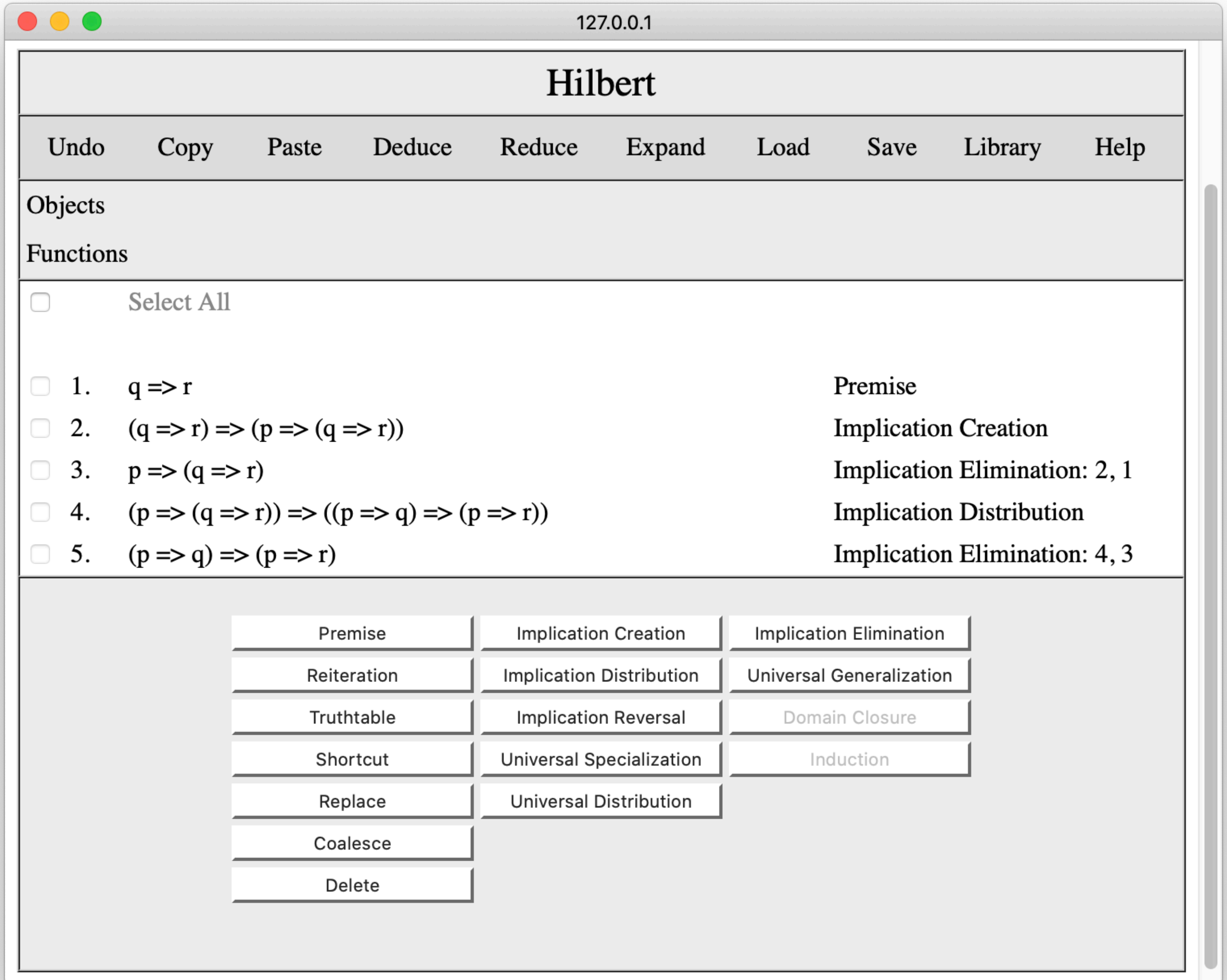
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Hilbert
Undo
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Deduce
Reduce
Expand
Load
Save
Library
Help
Objects
Functions
Select All
1. q => r
Premise
2. (q => r) => (p => (q => r))
Implication Creation
3. p => (q => r)
Implication Elimination: 2, 1
4. (p => (q => r)) => ((p => q) => (p => r))
Implication Distribution
5. (p => q) => (p => r)
Implication Elimination: 4, 3
Premise
Implication Creation
Implication Elimination
Reiteration
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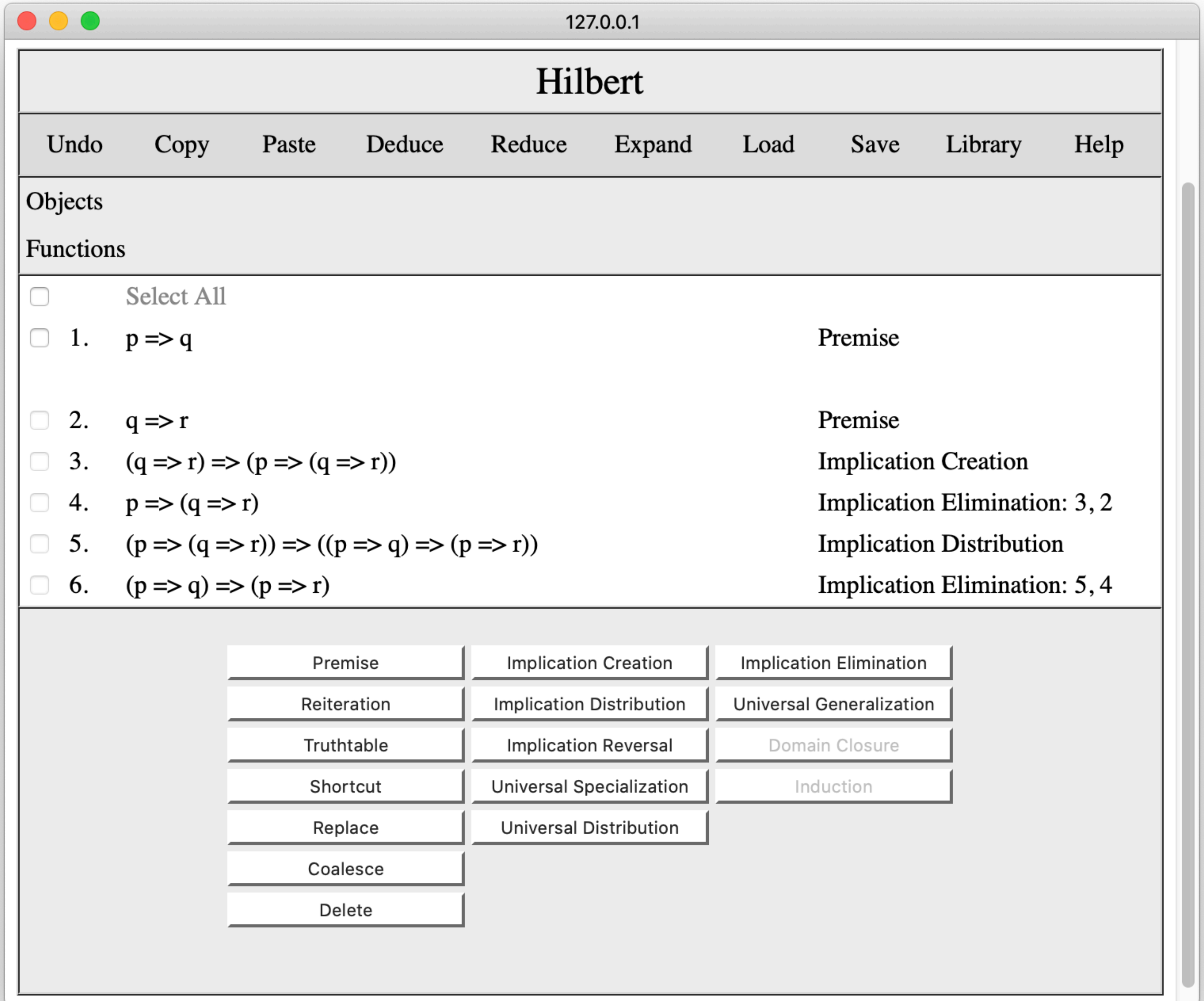
Universal Generalization
Truthtable
Implication Reversal
Domain Closure
Shortcut
Universal Specialization
Induction
Replace
Universal Distribution
Coalesce
Delete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Select All
1. p => q Premise
2. q => r Premise
3. (q => r) => (p => (q => r)) Implication Creation
4. p => (q => r) Implication Elimination: 3, 2
5. (p => (q => r)) => ((p => q) => (p => r)) Implication Distribution
6. (p => q) => (p => r) Implication Elimination: 5, 4
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Universal Distribution
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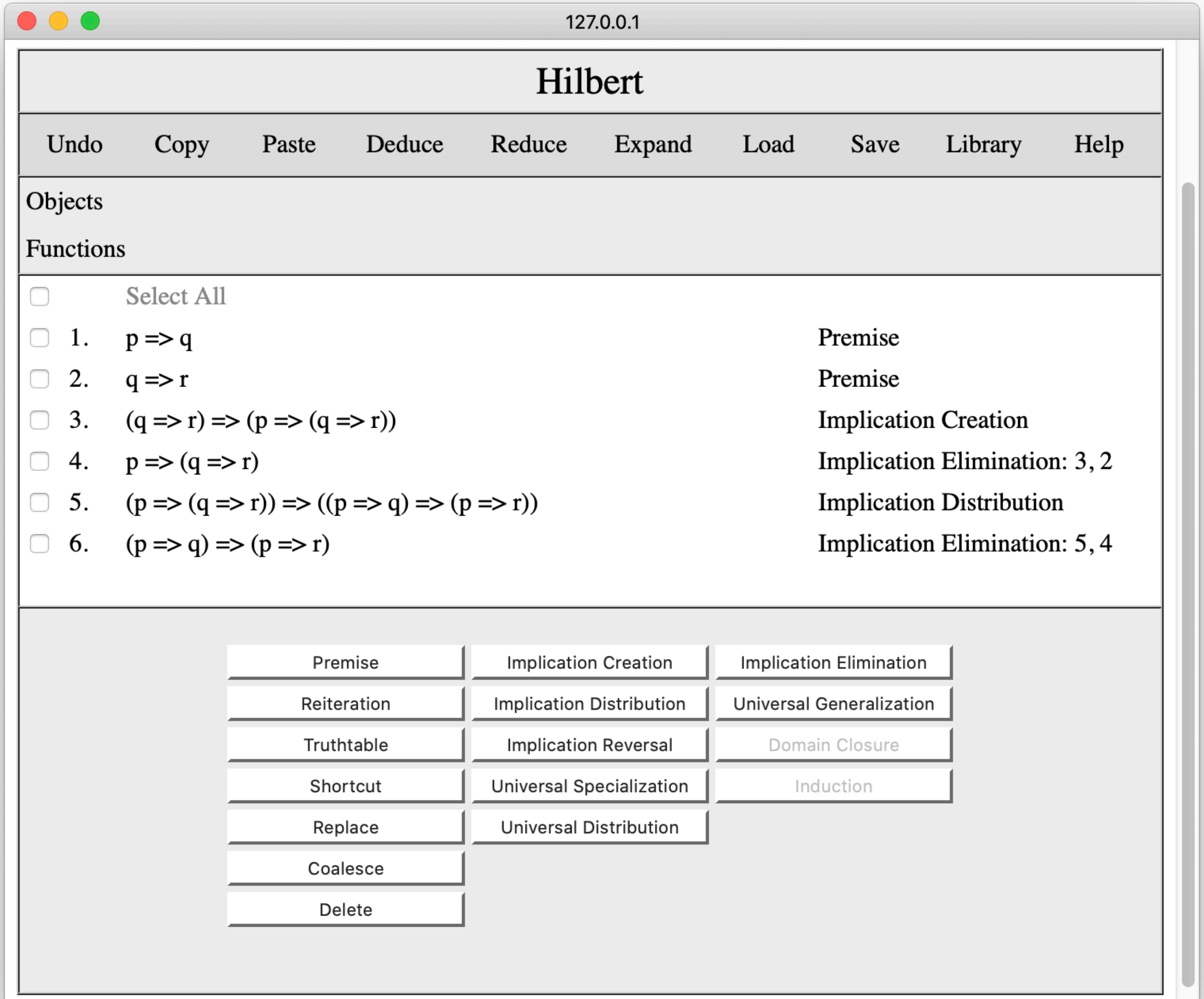
Delete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Objects
Functions
Select All
1. p => q Premise
2. q => r Premise
3. (q => r) => (p => (q => r)) Implication Creation
4. p => (q => r) Implication Elimination: 3, 2
5. (p => (q => r)) => ((p => q) => (p => r)) Implication Distribution
6. (p => q) => (p => r) Implication Elimination: 5, 4
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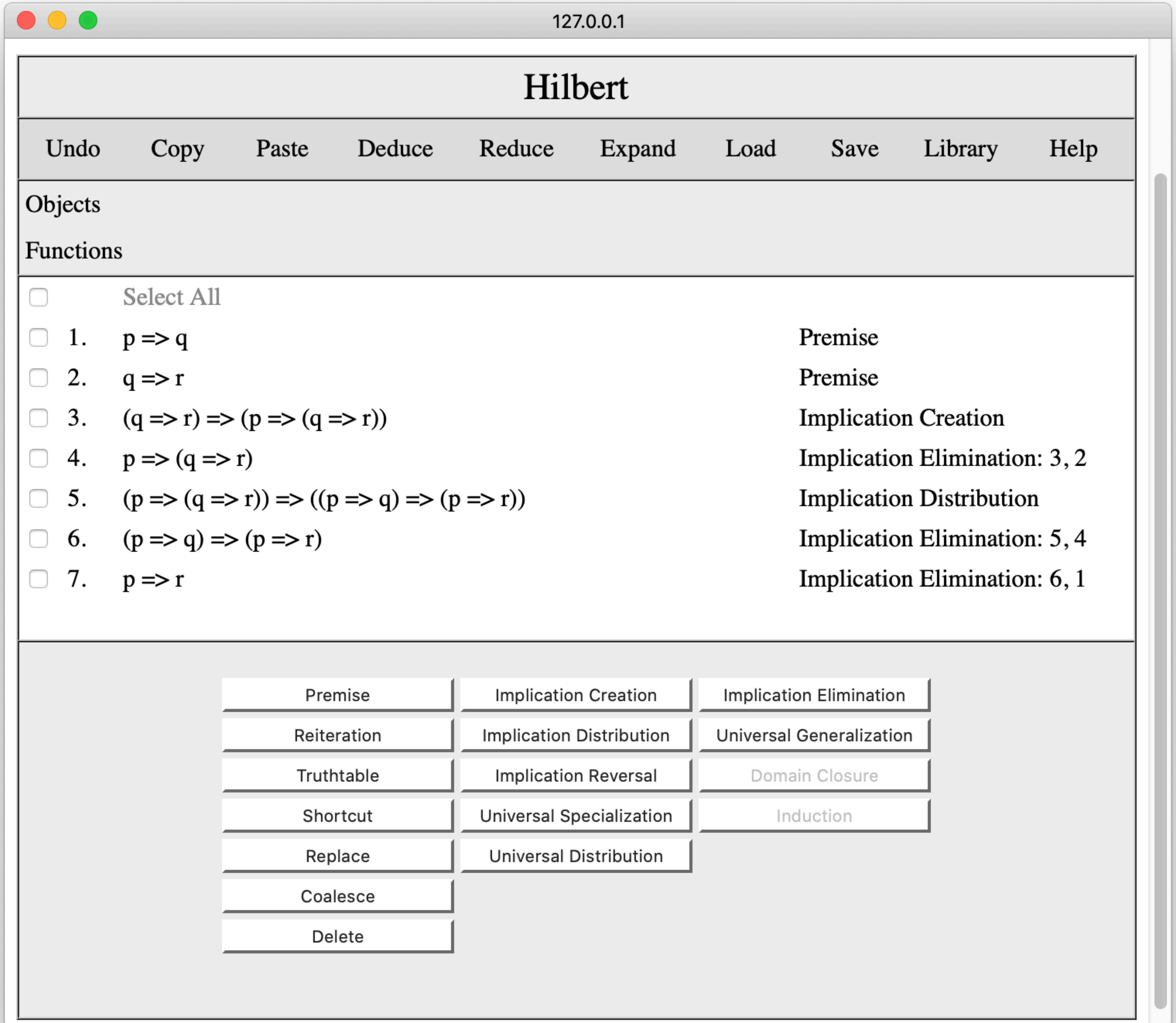
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7. p => r Implication Elimination: 6, 1
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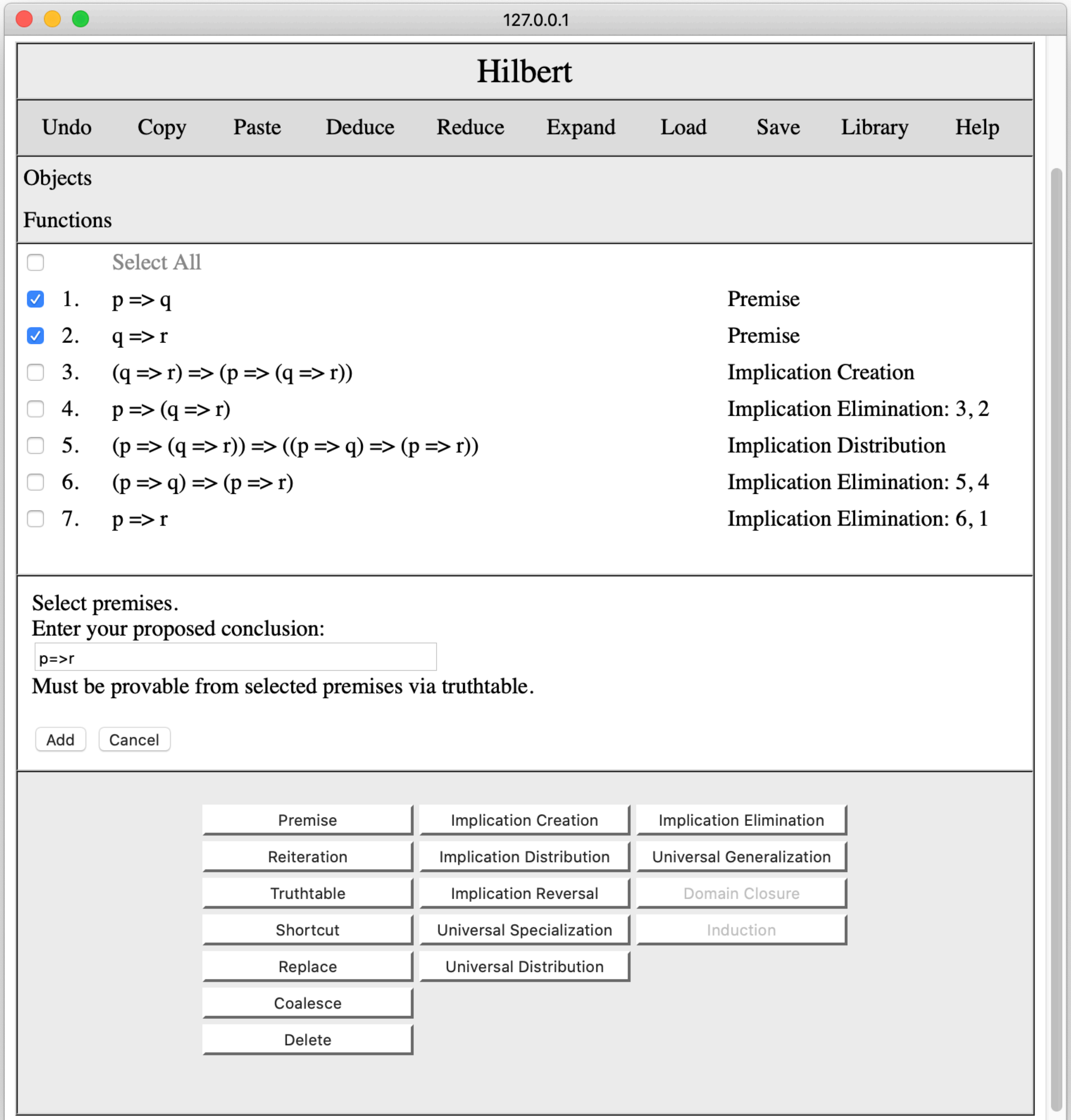

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1. p => q Premise
2. q => r Premise
3. (q => r) => (p => (q => r)) Implication Creation
4. p => (q => r) Implication Elimination: 3, 2
5. (p => (q => r)) => ((p => q) => (p => r)) Implication Distribution
6. (p => q) => (p => r) Implication Elimination: 5, 4
7. p => r Implication Elimination: 6, 1
Select premises.
Enter your proposed conclusion:
p=>r
Must be provable from selected premises via truthtable.
Add
Cancel
Premise
Implication Creation
Implication Elimination
Reiteration
Implication Distribution
Universal Generalization
Truthtable
Implication Reversal
Domain Closure
Shortcut
Universal Specialization
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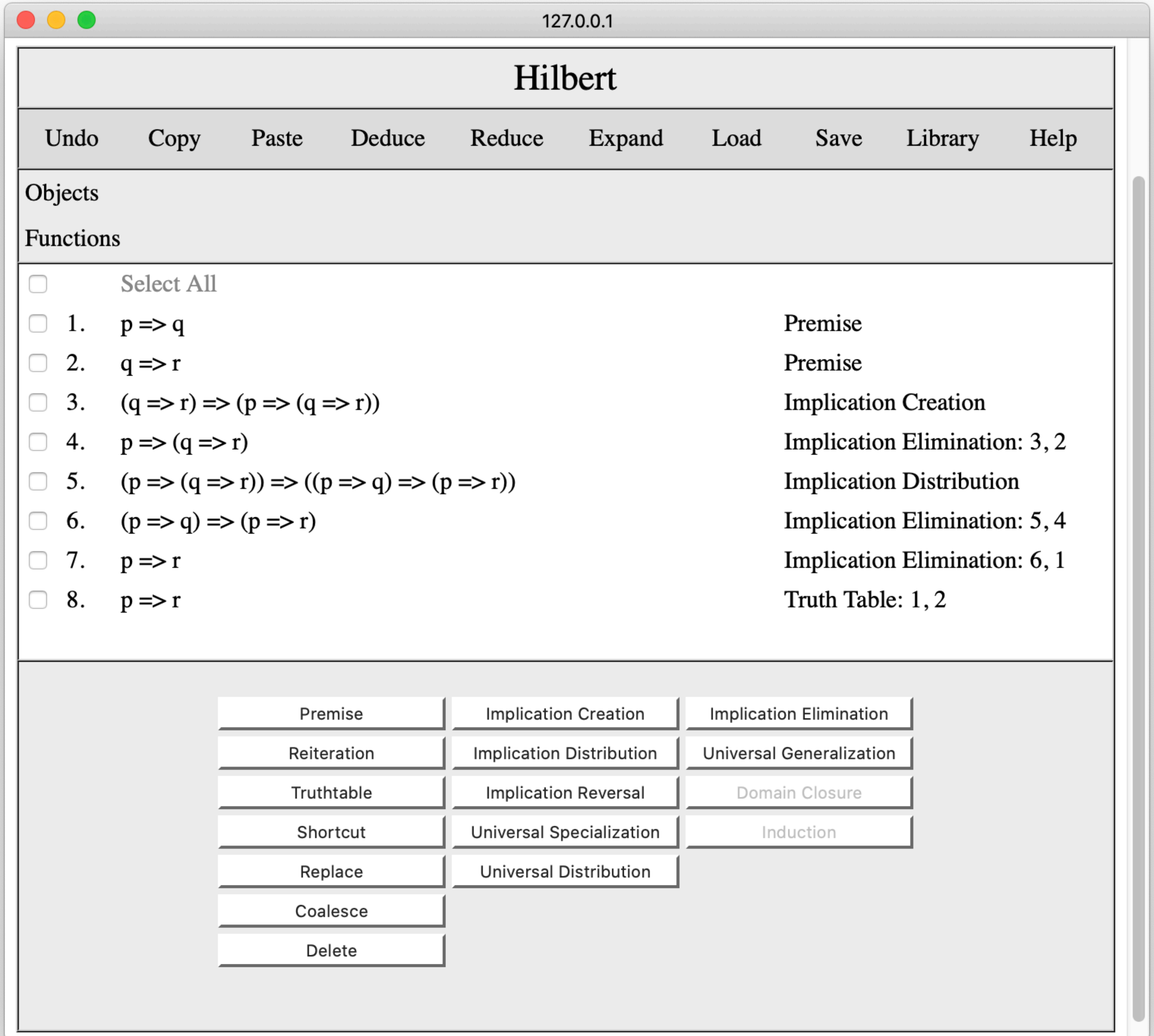
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6. (p => q) => (p => r) Implication Elimination: 5, 4
7. p => r Implication Elimination: 6, 1
8. p => r Truth Table: 1, 2
Premise
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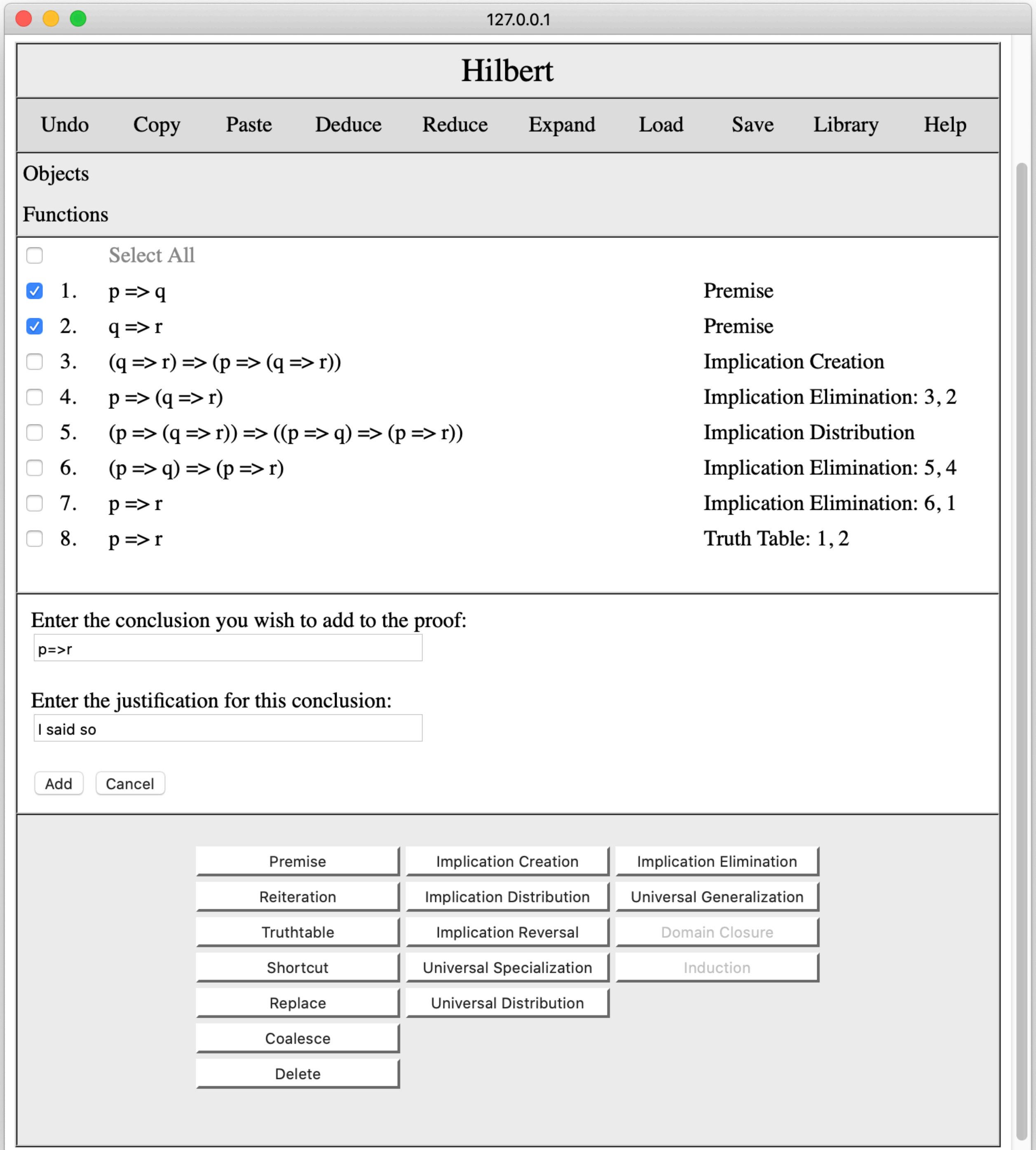
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5. (p => (q => r)) => ((p => q) => (p => r)) Implication Distribution
6. (p => q) => (p => r) Implication Elimination: 5, 4
7. p => r Implication Elimination: 6, 1
8. p => r Truth Table: 1, 2
Enter the conclusion you wish to add to the proof:
p=>r
Enter the justification for this conclusion:
I said so
Add
Cancel
Premise
Implication Creation
Implication Elimination
Reiteration
Implication Distribution
Universal Generalization
Truthtable
Implication Reversal
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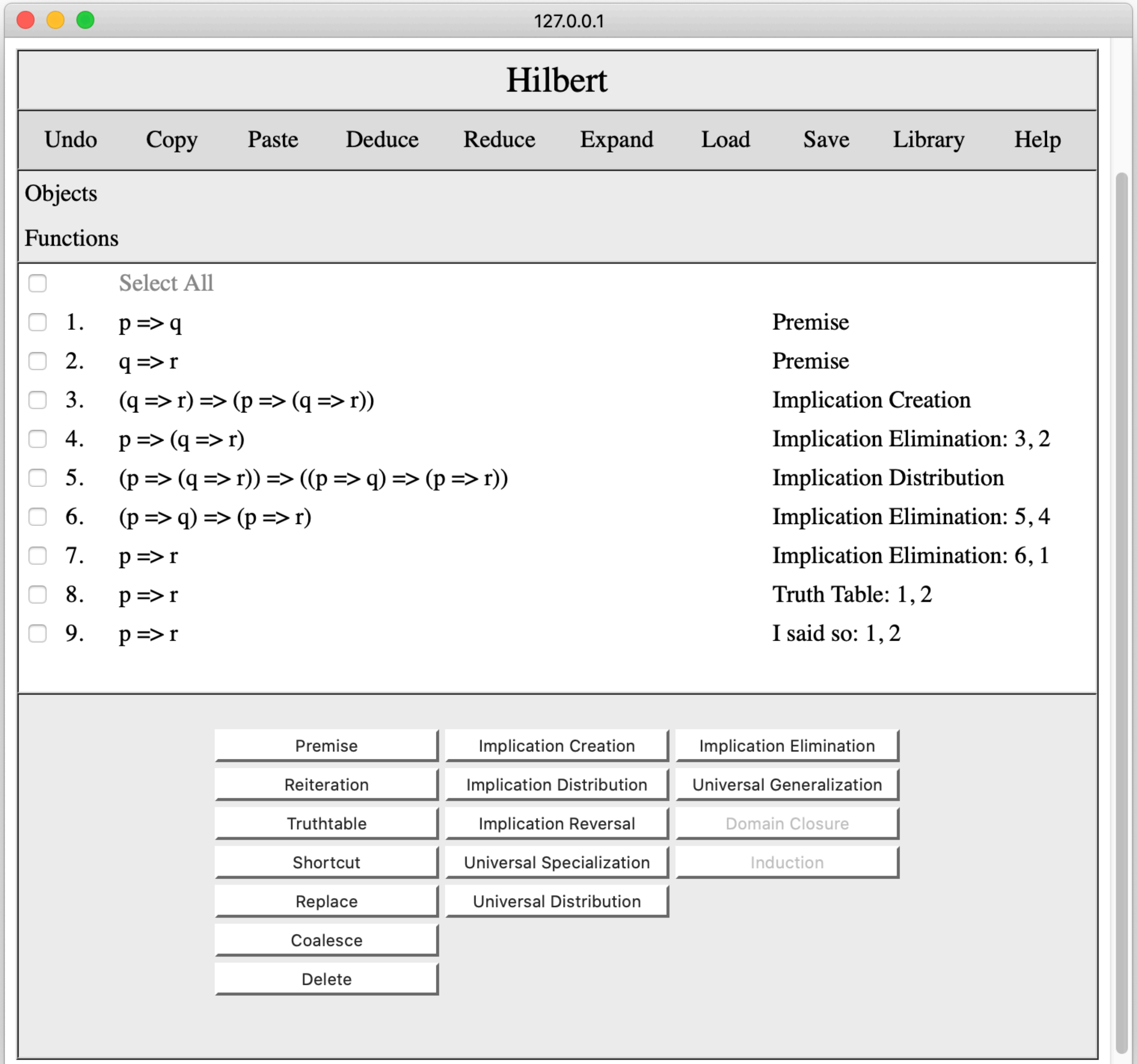
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8. p => r Truth Table: 1, 2
9. p => r I said so: 1, 2
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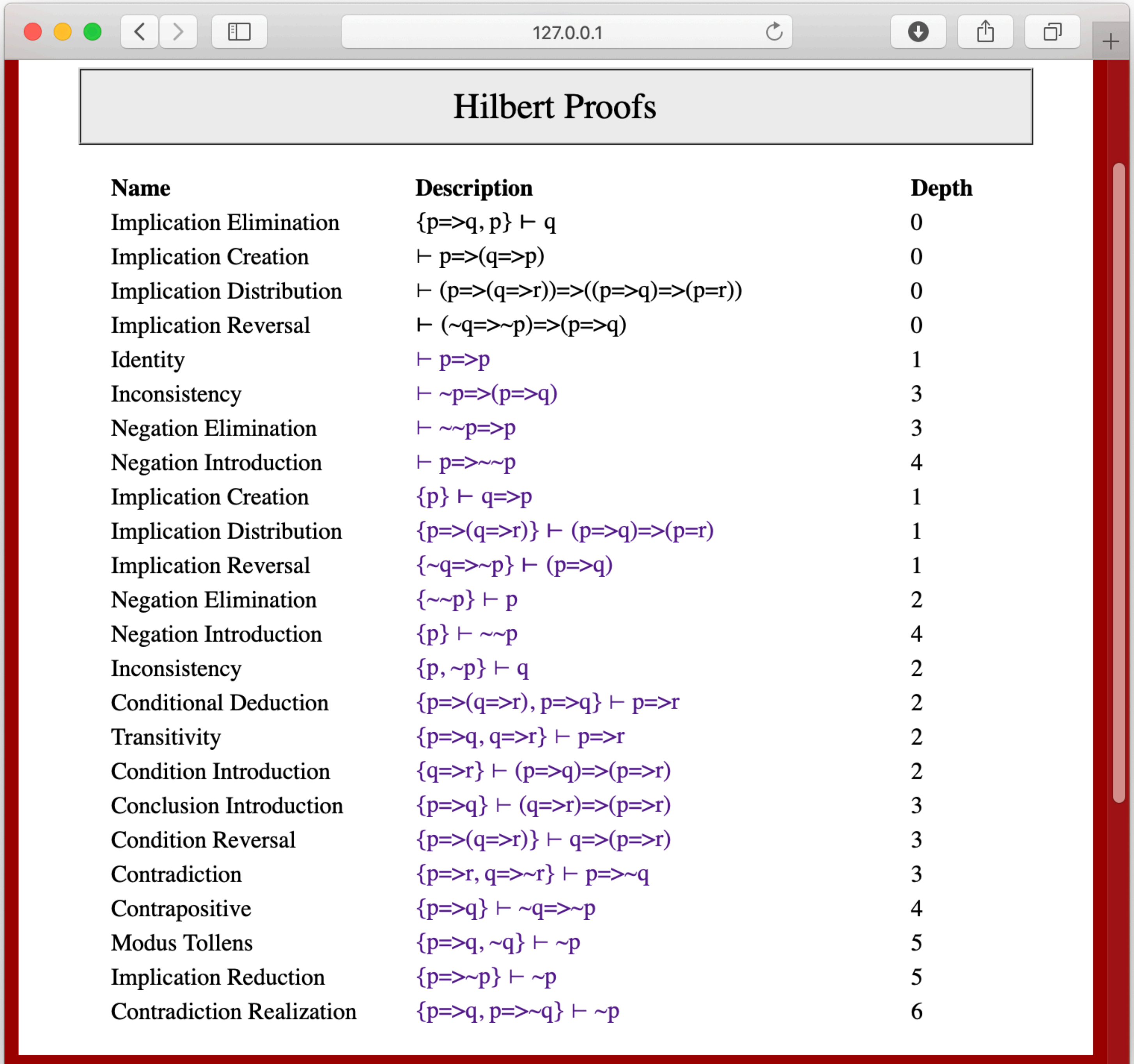

Hilbert Proofs

Name	Description	Depth
Implication Elimination	{p=>q, p} ⊢ q	0
Implication Creation	⊢ p=>(q=>p)	0
Implication Distribution	⊢ (p=>(q=>r))=>((p=>q)=>(p=r))	0
Implication Reversal	⊢ (~q=>~p)=>(p=>q)	0
Identity	⊢ p=>p	1
Inconsistency	⊢ ~p=>(p=>q)	3
Negation Elimination	⊢ ~~p=>p	3
Negation Introduction	⊢ p=>~~p	4
Implication Creation	{p} ⊢ q=>p	1
Implication Distribution	{p=>(q=>r)} ⊢ (p=>q)=>(p=r)	1
Implication Reversal	{~q=>~p} ⊢ (p=>q)	1
Negation Elimination	{~~p} ⊢ p	2
Negation Introduction	{p} ⊢ ~~p	4
Inconsistency	{p, ~p} ⊢ q	2
Conditional Deduction	{p=>(q=>r), p=>q} ⊢ p=>r	2
Transitivity	{p=>q, q=>r} ⊢ p=>r	2
Condition Introduction	{q=>r} ⊢ (p=>q)=>(p=>r)	2
Conclusion Introduction	{p=>q} ⊢ (q=>r)=>(p=>r)	3
Condition Reversal	{p=>(q=>r)} ⊢ q=>(p=>r)	3
Contradiction	{p=>r, q=>~r} ⊢ p=>~q	3
Contrapositive	{p=>q} ⊢ ~q=>~p	4
Modus Tollens	{p=>q, ~q} ⊢ ~p	5
Implication Reduction	{p=>~p} ⊢ ~p	5
Contradiction Realization	{p=>q, p=>~q} ⊢ ~p	6

Hilbert

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Objects

Functions

☐		Select All	
☐	1.	p => q	Premise
☐	2.	q => r	Premise
☐	3.	p => (q => r)	Implication Creation: 2
☐	4.	(p => q) => (p => r)	Implication Distribution: 3
☐	5.	p => r	Implication Elimination: 4, 1

Hilbert

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Objects

Functions

☐		Select All	
☐	1.	p	Premise
☐	2.	p => (q => p)	Implication Creation Schema
☐	3.	q => p	Implication Elimination: 2, 1

Hilbert

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Objects

Functions

☐		Select All	
☐	1.	p => (q => r)	Premise
☐	2.	(p => (q => r)) => ((p => q) => (p => r))	Implication Distribution Schema
☐	3.	(p => q) => (p => r)	Implication Elimination: 2, 1

From Hilbert to Fitch

Example

Example: Given $(p \Rightarrow q)$ and $(q \Rightarrow r)$, prove $(p \Rightarrow r)$.

1. $p \Rightarrow q$
2. $q \Rightarrow r$
3. $(q \Rightarrow r) \Rightarrow (p \Rightarrow (q \Rightarrow r))$
4. $(p \Rightarrow (q \Rightarrow r))$
5. $(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$
6. $(p \Rightarrow q) \Rightarrow (p \Rightarrow r)$
7. $p \Rightarrow r$

Deduction Theorem

Deduction Theorem: $\Delta \models (\varphi \Rightarrow \psi)$ if and only if $\Delta \cup \{\varphi\} \models \psi$.

Corollary: $\Delta \vdash (\varphi \Rightarrow \psi)$ if and only if $\Delta \cup \{\varphi\} \vdash \psi$.

Problem: Given $(p \Rightarrow q)$ and $(q \Rightarrow r)$, prove $(p \Rightarrow r)$.

Equivalent: Given $(p \Rightarrow q)$ and $(q \Rightarrow r)$ and p, prove r.

Deduction Theorem Example

Example: Given $(p \Rightarrow q)$ and $(q \Rightarrow r)$, prove $(p \Rightarrow r)$.

1. $p \Rightarrow q$
2. $q \Rightarrow r$
3. $(q \Rightarrow r) \Rightarrow (p \Rightarrow (q \Rightarrow r))$
4. $(p \Rightarrow (q \Rightarrow r))$
5. $(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$
6. $(p \Rightarrow q) \Rightarrow (p \Rightarrow r)$
7. $p \Rightarrow r$

1. $p \Rightarrow q$
2. $q \Rightarrow r$
3. p
4. q
5. r

Possible to derive long proof from short proof automatically.
Basis for the Fitch procedure.