Computational Logic Lecture 15

Equality

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Coreferentiality

Different terms can sometimes refer to the same object.

2+2 2*2 4

In some situations, we want to say explicitly that two terms are coreferential; and, in some situations, we want to say explicitly that two terms are not coreferential. For this, we use equality.

$$2+2=2*2$$
$$2+2\neq 2*3$$

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Equality

An equation $\sigma = \tau$ is true in an interpretation i if and only if the terms in the equation refer to the same object in the universe of discourse.

$$\sigma^i = \tau^i$$

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Example

Interpretation:

$i(a) = ○$
$i(b) = ●$
$i(c) = ○$
$i(f) = \{○\rightarrow●, ●\rightarrow○\}$
$i(r) = \{\langle○, ●\rangle, \langle●, ●\rangle\}$

Satisfied Sentences

$a = a$	$a = f(b)$	$a = f(f(a))$
$a \neq b$	$b = f(a)$	$a = f(f(c))$
$a = c$	$b = f(c)$	$b = f(f(b))$
$b = b$	$c = f(b)$	$c = f(f(a))$
$b \neq c$		$c = f(f(c))$

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Unique Names Assumption

In many applications, one makes the assumption that every distinct name refers to a distinct object, i.e. every object has at most one name. This is called the *unique names assumption* (UNA). The upshot is that a difference in name implies a difference in referent.

$$\sigma = \tau \text{ if and only if } \sigma^i = \tau^i$$

The unique names assumption is *not* true in general!!!

Question: How does one express the unique names assumption in Relational Logic?

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Domain Closure Assumption

In many applications, one makes the assumption that all objects are named, i.e. every object has at least one name. This is called the *domain closure assumption* (DCA).

The domain closure assumption is *not* true in general!!!

Question: How does one express the domain closure assumption in Relational Logic?

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Herbrand

The Herbrand Theorem does not hold in the presence of equality. In general, different constants *may* refer to the same object. In a Herbrand interpretation, every constant refers to itself. Thus, the following sentence is satisfiable, but there is no Herbrand interpretation that satisfies it.

$$john=jack$$

Question: Does the Herbrand theorem hold in the presence of the unique names assumption?

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Incompleteness

Theorem: Resolution (with factoring) is refutation complete for Relational Logic*.
*without equality.

Theorem: There is a set of premises Δ and a conclusion φ (involving equality) such that Δ logically implies φ but φ cannot be proved from Δ using Resolution.

1. $b = a$ Premise
2. $b = c$ Premise
3. $a = c$? Goal

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Two Approaches

Axiomatic approach -- add axioms of equality

Proof Theoretic approach -- add new rules of inference

Equality Axioms

Reflexivity

$$\forall x. x=x$$

Symmetry:

$$\forall x. \forall y. (x=y \Rightarrow y=x)$$

Transitivity:

$$\forall x. \forall y. \forall z. (x=y \wedge y=z \Rightarrow x=z)$$

Equality Axioms in Rule Form

Reflexivity

$$x=x$$

Symmetry:

$$x=y \Leftarrow y=x$$

Transitivity:

$$x=z \Leftarrow x=y \wedge y=z$$

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Equality Proof

1.	$b = a$	Premise
2.	$b = c$	Premise
3.	$x = x$	Equality
4.	$x = y \Leftarrow y = x$	Equality
5.	$x = z \Leftarrow x = y \wedge y = z$	Equality
6.	$a = c$?	Goal
7.	$a = y \wedge y = c$?	5,6
8.	$y = a \wedge y = c$?	4,7
9.	$b = c$?	1,8
10.	?	2,9

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Equality Problem

1.	$f(a) = b$	Premise
2.	$f(b) = a$	Premise
3.	$x = x$	Equality
4.	$x = y \Leftarrow y = x$	Equality
5.	$x = z \Leftarrow x = y \wedge y = z$	Equality
6.	$f(f(a)) = a?$	Goal
7.	$a = f(f(a))?$	4,6
8.	$f(f(a)) = y \wedge y = a?$	5,6
9.	$f(f(a)) = w \wedge w = y \wedge y = a?$	5,8
10.	$f(f(a)) = v \wedge v = w \wedge w = y \wedge y = a?$	5,9

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Flattening

Equivalence:

$$f(f(a))=a \Leftrightarrow \exists x.(f(a)=x \wedge f(x)=a)$$

Rewrite: $f(f(a))=a$
As: $\exists x.(f(a)=x \wedge f(x)=a)$
As: $f(a)=c \wedge f(c)=a$
As: $f(a)=c$
$f(c)=a$

Rewrite: $f(f(a))=a?$
As: $\exists x.(f(a)=x \wedge f(x)=a)?$
As: $f(a)=x \wedge f(x)=a?$

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Proof With Flattening

1.	$f(a) = b$	Premise
2.	$f(b) = a$	Premise
3.	$x = x$	Equality
4.	$x = y \Leftarrow y = x$	Equality
5.	$x = z \Leftarrow x = y \wedge y = z$	Equality
6.	$f(a) = x \wedge f(x) = a?$	$f(f(a)) = a?$
7.	$f(b) = a?$	1,6
8.	?	2,7

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Substitution Axiom

Flattening Rule:

$$f(f(a))=a \Leftrightarrow \exists x.(f(a)=x \wedge f(x)=a)$$

Substitution Axiom:

$$f(x)=z \Leftarrow x=y \wedge f(y)=z$$

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Proof With Substitution Axiom

1.	$f(a) = b$	Premise
2.	$f(b) = a$	Premise
3.	$x = x$	Equality
4.	$x = y \Leftarrow y = x$	Equality
5.	$x = z \Leftarrow x = y \wedge y = z$	Equality
6.	$f(x) = z \Leftarrow x = y \wedge f(y) = z$?	Substitution
7.	$f(f(a)) = a$?	Goal
8.	$f(a) = y \wedge f(y) = a$?	6,7
9.	$f(b) = a$?	1,8
10.	?	2,9

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Notes

Substitution axioms for relation constants too.

$$p(x) \Leftarrow x=y \wedge p(y)$$

Substitution axioms for multiple arguments

$$p(x,y)=z \Leftarrow x=u \wedge y=v \wedge p(u,v)$$

Need one substitution for each function and relation constant.

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Two Approaches

Axiomatic approach -- add axioms of equality

Proof Theoretic approach -- add new rules of inference

Motivation for Demodulation

$$\frac{\begin{array}{l} p(a, f(b, g(a, h(b)), c), d) \\ b = e \end{array}}{p(a, f(e, g(a, h(e)), c), d)}$$

Demodulation

$$\frac{\begin{array}{l}\{\varphi_1,\ldots,\varphi_n\}\\ \{\tau_1 = \tau_2\}\end{array}}{\{\varphi_1,\ldots,\varphi_n\}[\tau_1\sigma \leftarrow \tau_2\sigma]}$$

where τ occurs in φ_i

where $\tau_1\sigma = \tau$

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Examples

$$\frac{\begin{array}{l}p(a,f(b,g(a,h(b)),c),d)\\ b = e\end{array}}{p(a,f(e,g(a,h(e)),c),d)}$$

$$\frac{\begin{array}{l}p(a,f(b,g(a,h(b)),c),d)\\ g(x,y) = j(x)\end{array}}{p(a,f(b,j(a),c),d)}$$

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Non-Examples

Unit Equations Only

$$\frac{\begin{array}{l} p(a,g(a,b),c) \\ g(a,y)=y \Leftarrow p(y) \end{array}}{p(a,g(a,b),c) \Leftarrow p(b)}$$

Variables Substituted in Equation Only

$$\frac{\begin{array}{l} p(a,g(x,b),c) \\ g(a,y)=j(y) \end{array}}{p(a,j(b),c)}$$

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Parent Deletion

In Demodulation, parent is usually deleted after substitution.

1.	$a=1$	$a=1$
2.	$p(a,a,a)$	$p(a,a,a)$
3.	$p(a,a,1)$	$p(a,a,1)$
4.	$p(a,1,1)$	$p(a,1,a)$
5.	$p(1,1,1)$	$p(a,1,1)$
6.		$p(1,a,a)$
7.		$p(1,a,1)$
8.		$p(1,1,a)$
9.		$p(1,1,1)$

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Proof With Demodulation

1.	$f(a) = b$	Premise
2.	$f(b) = a$	Premise
3.	$x = x$	Equality
4.	$f(f(a)) = a?$	Goal
5.	$f(b) = a?$	1,4
6.	?	2,5

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Proof With Demodulation

1.	$f(1) = 1$	Premise
2.	$f(x) = x * f(x-1)$	Premise
3.	$x = x$	Equality
4.	*arithmetic axioms*	Premise
5.	$f(3) = z?$	Goal
6.	$3 * f(3-1) = z?$	2,5
7.	$3 * f(2) = z?$	4,6
8.	$3 * 2 * f(2-1) = z?$	2,7
9.	$3 * 2 * f(1) = z?$	4,8
10.	$3 * 2 * 1 = z?$	1,9
11.	$3 * 2 = z?$	4,10
12.	$6 = z?$	4,11
13.	?	3,12

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Proof With Demodulation

Suppose Bill is Joe's paternal grandfather and suppose that Bill is kind. Prove that Joe's paternal grandfather is kind.

1.	$pgf(joe) = bill$	Premise
2.	$f(f(x)) = pgf(x)$	Premise
3.	$kind(bill)$	Premise
4.	$kind(f(f(joe)))?$	Goal
5.	$kind(pgf(joe))?$	2,4
6.	$kind(bill)?$	1,5
7.	?	3,6

Problems With Demodulation

Cannot bind variables in expression:

$$\frac{\begin{array}{l} father(pat) = quincy \\ older(father(x), x) \end{array}}{older(quincy, pat)}$$

Equation must be a unit clause

$$\frac{\begin{array}{l} father(x) = y \Leftarrow x = pat \wedge y = quincy \\ older(father(x), x) \end{array}}{older(quincy, pat) \Leftarrow x = pat \wedge y = quincy}$$

Paramodulation

$$\frac{\begin{array}{l}\{\varphi_1,\ldots,\varphi_n\}\\ \{\psi_1,\ldots,\tau_1=\tau_2,\ldots,\psi_n\}\end{array}}{\{\varphi_1,\ldots,\varphi_n,\psi_1,\ldots,\psi_n\}\sigma[\tau_1\sigma\leftarrow\tau_2\sigma]}$$

where τ occurs in φ_i

where $\tau_1\sigma = \tau\sigma$

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Differences

(1) Demodulation requires a unit equation.

(2) Demodulation binds variables in equation only.

(3) Demodulation deletes parent.

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Example

$$\frac{\{p(f(x,b),x),q(x)\} \quad \{f(a,y)=y,r(y)\}}{\{p(b,a),q(a),r(b)\}}$$

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Proof With Paramodulation

1. $\{father(pat)=quincy\}$ Premise
2. $\{older(father(x),x)\}$ Premise
3. $\{\neg older(quincy,pat)\}$ Goal
4. $\{older(quincy,pat)\}$ 1,2
5. $\{\}$ 3,4

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Proof With Paramodulation

1.	$\{p(a)\}$	Premise
2.	$\{p(b)\}$	Premise
3.	$\{a = c, b = c\}$	$a = c \vee b = c$
4.	$\{\neg p(c)\}$	Goal
5.	$\{\neg p(a), b = c\}$	3,4
6.	$\{\neg p(a), \neg p(b)\}$	4,5
7.	$\{\neg p(b)\}$	1,6
8.	$\{\}$	2,7

Power

Theorem: Resolution and Paramodulation (together with the reflexivity axiom) are refutation complete for all of Relational Logic*.

*including equality