Computational Logic — Lecture 8

Relational Proofs

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Logical Entailment

A set of premises logically entails a conclusion if and only if every interpretation that satisfies the premises also satisfies the conclusion.

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Propositional Interpretations

	p	*q*	*r*
	0	0	0
	0	0	1
→	0	1	0
	0	0	1
	1	0	0
	1	0	1
	1	1	0
	1	1	1

For a language with n constants, there are 2^n interpretations.

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Relational Interpretations

$\|i\|$	*a*	*b*	*r*
{○,●}	○	○	{}
{○,●}	○	○	{○}
{○,●}	○	○	{●}
{○,●}	○	○	{○, ●}
{○,●}	○	●	{}
{○,●}	○	●	{○}
{○,●}	○	●	{●}
{○,●}	○	●	{○, ●}
{○,●}	●	○	{}
{○,●}	●	○	{○}
{○,●}	●	○	{●}
{○,●}	●	○	{○, ●}

. . .

Infinitely many interpretations.

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Logical Entailment and Provability

Good News: If Δ logically entails φ, then there is a finite proof of φ from Δ. And vice versa.

More Good News: If Δ logically entails φ, it is possible to find such a proof in finite time.

Sad News: If Δ does not logically entail φ, the process of finding a proof may run forever.

Not So Bad News: In many cases, the process can be stopped after finitely many steps.

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Formal Proofs

A *formal proof* of φ from Δ is a sequence of sentences terminating in φ in which each item is either:

1. a premise (a member of Δ)
2. an instance of an axiom schema
3. the result of applying a rule of inference to earlier items in the sequence.

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Old Rules of Inference

Modus Ponens (MP)

$$\frac{\varphi \Rightarrow \psi \quad \varphi}{\psi}$$

Modus Tolens (MT)

$$\frac{\varphi \Rightarrow \psi \quad \neg \psi}{\neg \varphi}$$

And Introduction (AI)

$$\frac{\varphi \quad \psi}{\varphi \wedge \psi}$$

And Elimination (AE)

$$\frac{\varphi \wedge \psi}{\varphi \quad \psi}$$

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Universal Generalization

Rule of Inference

$$\frac{\varphi}{\forall \nu . \varphi}$$

Examples:

$$\frac{p(x)}{\forall x . p(x)}$$

$$\frac{p(x) \Rightarrow q(x)}{\forall x . (p(x) \Rightarrow q(x))}$$

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Existential Generalization

Rule of Inference

$$\frac{\varphi[\tau]}{\exists \nu.\varphi[\nu]}$$

Examples:

$$\frac{p(a)}{\exists x.p(x)} \qquad \frac{p(a) \vee q(a)}{\exists x.(p(x) \vee q(x))}$$

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Idea for Universal Instantiation

$$\frac{\forall \nu.\varphi}{\varphi[\nu \leftarrow \tau]}$$

Warning: This is not quite right.

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Examples

$\forall y.hates(jane,y)$

$hates(jane,jill)$	$y \leftarrow jill$
$hates(jane,mother(jane))$	$y \leftarrow mother(jane)$
$hates(jane,y)$	$y \leftarrow y$
$hates(jane,z)$	$y \leftarrow z$

$\forall x.\exists y.hates(x,y)$

$\exists y.hates(jane,y)$	$x \leftarrow jane$	
$\exists y.hates(y,y)$	$x \leftarrow y$	*Wrong!!*

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Bounding

A term τ is *bound* by ν in φ if and only if τ contains a variable μ *and* there is some free occurrence of ν in φ *and* that occurrence lies in the scope of a quantifier of μ.

mother*(*x*)** is bound by ***y in **∃*x.hates*(*x,y*)**.

Why?

The term *mother(x)* contains a variable *x*.
There is a free occurrence of *y* in $\exists x.hates(x,y)$.
That occurrence of *y* lies in scope of quantifier of *x*.

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Substitutability

A term τ is *substitutable* for ν in φ if and only if it is *not* bound by ν in φ.

Some texts say "x is *free* for y in φ" instead of "x is substitutable for y in φ".

mother(*jane*) is free for y in *hates*(*jane*,y).
mother(x) is free for y in *hates*(*jane*,y).
mother(x) is free for y in $\exists z.hates(z,y)$.
mother(x) is *not* free for y in $\exists x.hates(x,y)$.
mother(x) is free for y in $(\forall x.\forall y.l(x,y) \wedge \exists z.h(z,y))$.

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Inappropriateness

An occurrence of a term τ is *inappropriate* for a variable ν in φ if and only if τ contains a variable μ *and* there is some free occurrence of ν in φ that lies in the scope of a quantifier of μ.

mother(x) is inappropriate for y in $\exists x.hates(x,y)$.

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Substitutability

A term τ is *substitutable* for ν in φ if and only if it is not inappropriate with ν in φ.

Some texts say "x is *free* for y in φ" instead of "x is substitutable for y in φ".

mother(*jane*) is free for y in *hates*(*jane*,y).
mother(x) is free for y in *hates*(*jane*,y).
mother(x) is free for y in $\exists z$.*hates*(z,y).
mother(x) is *not* free for y in $\exists x$.*hates*(x,y).
mother(x) is free for y in $(\forall x.\forall y.l(x,y) \wedge \exists z.h(z,y))$.

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Universal Instantiation

$$\frac{\forall \nu.\varphi}{\varphi[\nu \leftarrow \tau]}$$

where τ is free for ν in φ

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Existential Instantiation I

$$\frac{\exists \nu.\varphi}{\varphi[\nu \leftarrow \sigma]}$$

where $\exists \nu.\varphi$ contains no free variables

where σ is a new object constant

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Examples

$\exists y.p(y)$

$p(c)$

$\exists y.y*y{=}0$

1*1=0 *Wrong!*

$\exists y.y*y{=}x$

$c*c{=}x$ *Wrong!*

$c*c{=}4$

$c*c{=}6$

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Existential Instantiation II

$$\frac{\exists \nu.\varphi}{\varphi[\nu \leftarrow \pi(\tau_1,...,\tau_n)]}$$

where $\tau_1,...,\tau_n$ are free in $\exists \nu.\varphi$

where π is a new function constant

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Examples

$\exists y.y*y=x$

$f(x)*f(x)=x$

$f(4)*f(4)=4$

$f(6)*f(6)=6$

$\exists y.y*y=x$

$sqrt(x)*sqrt(x)=x$

$log(x)*log(x)=x$ *Wrong!*

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Formal Proofs

A *formal proof* of φ from Δ is a sequence of sentences terminating in φ in which each item is either:

1. a premise (a member of Δ)
2. an instance of an axiom schema
3. the result of applying a rule of inference to earlier items in the sequence.

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Example

Everybody loves somebody. Everybody loves a lover. Show that Jack loves Jill.

1.	$\forall x.\exists y.loves(x, y)$	Premise
2.	$\forall u.\forall v.\forall w.(loves(v, w) \Rightarrow loves(u, v))$	Premise
3.	$\exists y.loves(jill, y)$	UI : 1
4.	$loves(jill, mike)$	EI : 3
5.	$\forall v.\forall w.(loves(v, w) \Rightarrow loves(jack, v))$	UI : 2
6.	$\forall w.(loves(jill, w) \Rightarrow loves(jack, jill))$	UI : 5
7.	$loves(jill, mike) \Rightarrow loves(jack, jill)$	UI : 6
8.	$loves(jack, jill)$	MP : 7, 4

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Harry and Ralph

Every horse can outrun every dog. Some greyhounds can outrun every rabbit. Harry is a horse. Ralph is a rabbit. Can Harry outrun Ralph?

1.	$\forall x.\forall y.(h(x) \wedge d(y) \Rightarrow f(x,y))$	Premise
2.	$\exists y.(g(y) \wedge \forall z.(r(z) \Rightarrow f(y,z)))$	Premise
3.	$\forall y.(g(y) \Rightarrow d(y))$	Premise
4.	$\forall x.\forall y.\forall z.(f(x,y) \wedge f(y,z) \Rightarrow f(x,z))$	Premise
5.	$h(harry)$	Premise
6.	$r(ralph)$	Premise

Harry and Ralph (continued)

7.	$g(greg) \wedge \forall z.(r(z) \Rightarrow f(greg,z))$	$\mathrm{EI}:2$
8.	$g(greg)$	$\mathrm{AE}:7$
9.	$\forall z.(r(z) \Rightarrow f(greg,z))$	$\mathrm{AE}:7$
10.	$r(ralph) \Rightarrow f(greg,ralph)$	$\mathrm{UI}:9$
11.	$f(greg,ralph)$	$\mathrm{MP}:6,10$
12.	$g(greg) \Rightarrow d(greg)$	$\mathrm{UI}:3$
13.	$d(greg)$	$\mathrm{MP}:12,8$
14.	$\forall y.(h(harry) \wedge d(y) \Rightarrow f(harry,y))$	$\mathrm{UI}:1$
15.	$h(harry) \wedge d(greg) \Rightarrow f(harry,greg)$	$\mathrm{UI}:14$
16.	$h(harry) \wedge d(greg)$	$\mathrm{AI}:5,13$
17.	$f(harry,greg)$	$\mathrm{MP}:15,16$

Harry and Ralph (continued)

18.	$\forall y.\forall z.(f(harry,y) \wedge f(y,z) \Rightarrow f(harry,z))$	UI : 4
19.	$\forall z.(f(harry,greg) \wedge f(greg,z) \Rightarrow f(harry,z))$	UI : 18
20.	$f(harry,greg) \wedge f(greg,ralph) \Rightarrow f(harry,ralph))$	UI : 19
21.	$f(harry,greg) \wedge f(greg,ralph)$	AI : 17,11
22.	$f(harry,ralph)$	MP : 20, 21

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Mendelson Logic

Mendelson Logic is that subset of Relational Logic in which there are only two operators, viz. $\neg$ and $\Rightarrow$, and one quantifier, viz. $\forall$. Fortunately, all sentences in Relational Logic can be reduced to logically equivalent sentences with these operators by applying the following rules.

$$(\psi \Leftrightarrow \varphi) \rightarrow ((\varphi \Rightarrow \psi) \wedge (\psi \Rightarrow \varphi))$$
$$(\varphi \Leftarrow \psi) \rightarrow (\psi \Rightarrow \varphi)$$
$$(\psi \wedge \varphi) \rightarrow \neg(\neg\varphi \Rightarrow \psi)$$
$$(\psi \vee \varphi) \rightarrow (\neg\varphi \Rightarrow \psi)$$
$$\exists \nu.\varphi \rightarrow \neg\forall \nu.\neg\varphi$$

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Mendelson Rules of Inference

Modus Ponens (MP)

$$\frac{\varphi \Rightarrow \psi \qquad \varphi}{\psi}$$

Universal Generalization (UG)

$$\frac{\varphi}{\forall \nu . \varphi}$$

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Mendelson Axiom Schemata

II: $\varphi \Rightarrow (\psi \Rightarrow \varphi)$

ID: $(\varphi \Rightarrow (\psi \Rightarrow \chi)) \Rightarrow ((\varphi \Rightarrow \psi) \Rightarrow (\varphi \Rightarrow \chi))$

CR: $(\neg\psi \Rightarrow \varphi) \Rightarrow ((\neg\psi \Rightarrow \neg\varphi) \Rightarrow \psi)$
$(\psi \Rightarrow \varphi) \Rightarrow ((\psi \Rightarrow \neg\varphi) \Rightarrow \neg\psi)$

UD: $\forall\nu.(\varphi \Rightarrow \psi) \Rightarrow (\forall\nu.\varphi \Rightarrow \forall\nu.\psi)$

UI: $\forall\nu.\varphi \Rightarrow \varphi[\nu \leftarrow \tau]$
where τ is free for ν in φ

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Provability

A sentence φ is *provable* from a set of sentences Δ if and only if there is a finite formal proof of φ from Δ using only Modus Ponens, Universal Generalization, and the Mendelson axiom schemata.

Soundness Theorem: If φ is provable from Δ, then Δ logically entails φ.

Completeness Theorem (Godel): If Δ logically entails φ, then φ is provable from Δ.

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Decidability

A *class* of questions is *decidable* if and only if there is a procedure such that, when given as input any question in the class, the procedure halts and says *yes* if the answer is positive and *no* if the answer is negative.

Example: For any natural number *n*, determining whether *n* is prime.

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Semidecidability

A *class* of questions is semidecidable if and only if there is a procedure that halts and says *yes* if the answer is positive.

Obvious Fact: If a class of questions is decidable, it is semidecidable.

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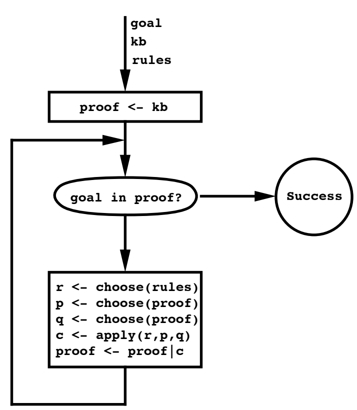
Semidecidability of Logical Entailment



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Decidability Not Proved

Note that we have *not* shown that logical entailment for Relational Logic is decidable.

The procedure may not halt.

$p(x) \Rightarrow p(f(x))$
$p(f(f(a)))$
$p(f(b))$?

We cannot just run procedure on negated sentence because that may not be logically entailed either!

$p(x) \Rightarrow p(f(x))$
$p(f(f(a)))$
$\neg p(f(b))$?

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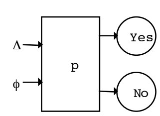
Undecidability of Logical Entailment

Metatheorem: Logical Entailment for Relational Logic is *not* decidable.

Proof: Suppose there is a machine *p* that decides the question of logical entailment. Its inputs are Δ and φ.



We can encode the behavior of this machine and its inputs as sentences and ask whether the machine halts as a conclusion.

What happens if we give this description and question to *p*? It says *yes*.

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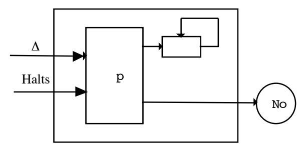
Undecidability (continued)

It is possible to construct a larger machine *p'* that enters an infinite loop if *p* says *yes* and halts if *p* says *no*.



We can also encode a description of this machine as a set of sentences and ask whether the machine halts as a conclusion.

What happens if we give this description and question to *p'*? If *p* says *yes*, then *p'* runs forever, contradicting the hypothesis that *p* computes correctly. If *p* says *no*, then *p'* halts, once again leading to contradiction. QED

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Closure

The *closure* S^* of a set S of sentences is the set of all sentences logically entailed by S.

$$S^* = \{\varphi \mid S \models \varphi\}$$

Set of Sentences:

$p(a)$
$p(x) \Rightarrow p(f(x))$

Closure:

$p(a)$
$p(f(a))$
$p(f(f(a)))$
$p(a) \wedge p(f(a))$
$p(x) \Rightarrow p(f(x))$
…

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Theories

A *theory* is a set of sentences closed under logical entailment, i.e. T is a theory if and only if $T^*=T$.

A theory T is *finitely axiomatizable* if and only if there is a *finite* set Δ of sentences such that $T=\Delta^*$.

A theory T is *complete* if and only, for all φ, either $\varphi \in T$ or $\neg\varphi \in T$.

Note: Not every theory is complete. Consider the theory consisting of all consequences of $p(a,b)$. Does this include $p(b,a)$? Does it include $\neg p(b,a)$?

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Relationships on Theories

Decidable

Semidecidable   Finitely Axiomatizable

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Arithmetization of Logical Entailment

The theory of arithmetic is the set of all sentences true of the natural numbers, 0, 1, +, *, and <.

Fact: It is possible to assign numbers to sentences such that
(1) Every sentence φ is assigned a unique number n_φ.
(2) The question of logical entailment $\Delta|=\varphi$ can be expressed as a numerical condition $r(n_\Delta,n_\varphi)$.

Conclusion: The theory of arithmetic is not decidable.

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Incompleteness Theorem

Metatheorem (Godel): If Δ is a finite subset of the theory of arithmetic, then Δ^* is not complete.

Variant: Arithmetic is not finitely axiomatizable.

Proof: If there were a finite axiomatization, then the theory would be decidable. However, arithmetic is not decidable. Therefore, there is no finite axiomatization.

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Summary

Logical Entailment for Relational Logic is semidecidable.

Logical Entailment for Relational Logic is *not* decidable.

Arithmetic is not finitely axiomatizable in Relational Logic.

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