

Relational Logic

Propositional Logic

Proposition constants refer to atomic propositions.

raining *snowing* *wet*

Compound sentences capture relationships among propositions.

$raining \vee snowing \Rightarrow wet$

Relational Logic

Object constants refer to objects.

john mary

Relation constants refer to relationships.

loves happy

Sentences express information about objects and their relationships.

$loves(joe,mary)$
 $loves(x,y) \Rightarrow loves(y,x)$
 $loves(x,y) \wedge loves(y,x) \Rightarrow happy(x)$

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Plan of Action

Relational Logic Syntax and Informal Semantics

Formal Semantics

Herbrand Method

Relational Proofs

Resolution Preliminaries

Resolution

Applications

Strategies

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Words

Variables begin with characters from the end of the alphabet (from *u* through *z*).

u, v, w, x, y, z

Constants begin with digits or letters from the beginning of the alphabet (from *a* through *t*).

a, b, c, arthur, betty, cathy, 1, 2, ...

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Constants

Object constants refer to objects.

joe, stanford, usa, 2345

Function constants denote functions.

father, mother, age, plus, times

Relation constants refer to relations.

person, happy, parent, loves

There is no syntactic distinction between object, function, and relation constants. The type of each such word is determined from context.

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Arity

The arity of a function constant or a relation constant is the number of arguments it takes.

Unary Function constants: *father, mother*

Binary Function constants: *plus, times*

Ternary Function constants: *price*

Unary Relation constants: *person, happy*

Binary Relation constants: *parent, loves*

Ternary Relation constants: *between*

The arity of a function constant or a relation constant is optionally notated as a subscript on the constant.

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Terms

A *term* is either a variable, an object constant, or a functional term.

Terms refer to items in the universe of discourse.

Terms are analogous to noun phrases in natural language.

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Functional Terms

A *functional term* is an expression formed from an n -ary function constant and n terms enclosed in parentheses and separated by commas.

father(joe)
age(joe)
plus(x,2)

Functional terms are terms and so can be nested.

plus(age(father(joe)),age(mother(joe)))

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Sentences

There are three types of sentences.

Relational sentences - analogous to the simple sentences in natural language

Logical sentences - analogous to the compound sentences in natural language

Quantified sentences - sentences that express the significance of variables

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Relational Sentences

A *relational sentence* is an expression formed from an n -ary relation constant and n terms enclosed in parentheses and separated by commas.

happy(art)
loves(art,cathy)

Relational sentences are *not* terms and *cannot* be nested in terms or relational sentences.

No! *happy(person(joe))* No!

happy(joe)
person(joe)

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Logical Sentences

Logical sentences in Relational Logic are analogous to those in Propositional Logic.

$\neg \text{loves}(\text{art}, \text{cathy})$
 $(\text{loves}(\text{art}, \text{betty}) \wedge \text{loves}(\text{betty}, \text{art}))$
 $(\text{loves}(\text{art}, \text{betty}) \vee \text{loves}(\text{art}, \text{cathy}))$
 $(\text{loves}(x, y) \Rightarrow \text{loves}(y, x))$
 $(\text{loves}(x, y) \Leftarrow \text{loves}(y, x))$
 $(\text{loves}(x, y) \Leftrightarrow \text{loves}(y, x))$

Parenthesization rules are the same as for Propositional Logic.

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Quantified Sentences

Universal sentences assert facts about all objects.

$$\forall x.(person(x) \Rightarrow mammal(x))$$

Existential sentence assert the existence of an object with given properties.

$$\exists x.(person(x) \wedge happy(x))$$

Quantified sentences can be nested within other sentences.

$$\forall x.apple(x) \vee \exists x.pear(x)$$

$$\forall x.\exists y.loves(x,y)$$

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Syntax Test

Object Constants: *art, betty, cathy*

Unary Function Constants: *father, mother, age*

Binary Function Constant: *plus*

Unary Relation Constants: *person, happy*

Binary Relation Constants: *reflexive, parent, loves*

loves(father(art),mother(art))

plus(father(art),betty)

happy(person(cathy))

$\forall x.\forall y.(loves(x,y) \Rightarrow loves(y,x))$

$loves(x,y) \Rightarrow loves(y,x)$

$\forall x.\forall y.loves(art,cathy)$

$\forall z.\forall x.reflexive(z) \Rightarrow z(x,x)$

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Reminder

Functional terms and relational sentences look similar. However, they are not the same.

Functional terms *may* be used within other functional terms. Functional terms *may* be used within relational sentences.

Relational sentences may *not* be used in functional terms. Relational sentences may *not* be used in relational sentences.

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Infix Syntax for Functions

$plus(2,3)$	\Leftrightarrow	$2 + 3$
$minus(2,3)$	\Leftrightarrow	$2 - 3$
$times(2,3)$	\Leftrightarrow	2×3
$quotient(2,3)$	\Leftrightarrow	$2 \div 3$
$expt(2,3)$	\Leftrightarrow	$2 \uparrow 3$
$union(s,t)$	\Leftrightarrow	$s \cup t$
$intersection(s,t)$	\Leftrightarrow	$s \cap t$

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Infix Syntax for Relations

$$eq(2,3) \leftrightarrow 2 = 3$$

$$nq(2,3) \leftrightarrow 2 \neq 3$$

$$lt(2,3) \leftrightarrow 2 < 3$$

$$gt(2,3) \leftrightarrow 2 > 3$$

$$leq(2,3) \leftrightarrow 2 \leq 3$$

$$geq(2,3) \leftrightarrow 2 \geq 3$$

$$member(s,t) \leftrightarrow s \in t$$

$$subset(s,t) \leftrightarrow s \subset t$$

$$subseteq(s,t) \leftrightarrow s \subseteq t$$

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Operator Precedence

↑

× ÷

+ -

∩

∪

= ≠ < > ≤ ≥

∈ ∉ ⊂ ⊃ ⊆ ⊇

¬ ∀ ∃

∧

∨

⇐ ⇔ ⇒

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Mushrooms

Unary relation constants: *mushroom*, *purple*, *poison*

Purple mushrooms are poison.

If a thing is a purple mushroom, then it is poison.

If a thing is mushroom and it is purple, then it is poison.

$$\forall x.(mushroom(x) \wedge purple(x) \Rightarrow poison(x))$$

No purple mushroom is poison.

There is nothing that is a mushroom and purple and poison.

$$\neg \exists x.(mushroom(x) \wedge purple(x) \wedge poisonous(x))$$

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More Mushrooms

Unary relation constants: *mushroom*, *purple*, *poison*

A mushroom is poison only if it is purple.

If a thing is a mushroom, it is poison only if it is purple.

If a thing is a mushroom and it is poison, then it is purple.

$$\forall x.(mushroom(x) \wedge poison(x) \Rightarrow purple(x))$$

A mushroom is not poison unless it is purple.

If a thing is a mushroom, it is not poison unless it is purple.

If a thing is a mushroom and it is not purple, then it is not poison.

$$\forall x.(mushroom(x) \wedge \neg purple(x) \Rightarrow \neg poison(x))$$

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Interpersonal Relations

Object constants: *mike, maureen*

Binary relation constant: *loves*

Everybody loves Maureen.

$\forall x.loves(x,maureen)$

Maureen loves everyone who loves her.

$\forall x.(loves(x,maureen) \Rightarrow loves(maureen,x))$

Nobody loves Mike.

$\neg \exists x.loves(x,mike)$

Nobody who loves Maureen loves Mike.

$\forall x.(loves(x,maureen) \Rightarrow \neg loves(x,mike))$

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More Interpersonal Relations

Object constants: *mike, maureen*

Binary relation constant: *loves*

Everybody loves somebody.

$\forall x.\exists y.loves(x,y)$

There is somebody whom everybody loves.

$\exists y.\forall x.loves(x,y)$

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Abelian Groups

Associativity Axiom

$$(x + y) + z = x + (y + z)$$

Commutativity Axiom

$$x + y = y + x$$

Identity Axioms

$$0 + y = y$$

$$y + 0 = y$$

Inverse Axioms

$$x + \text{inv}(x) = 0$$

$$\text{inv}(x) + x = 0$$

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Open Partial Orders

Non-reflexivity

$$\neg x < x$$

Asymmetry

$$x < y \Rightarrow \neg y < x$$

Transitivity

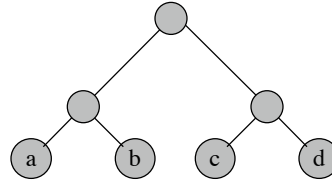
$$x < y \wedge y < z \Rightarrow x < z$$

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Binary Trees

Representation as a term:

$pair(pair(a,b),$
 $pair(c,d))$



Membership axioms:

$in(x,x)$

$in(x,pair(y,z)) \Leftarrow in(x,y) \vee in(x,z)$

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Variable Length Lists

Example

$[a,b,c,d]$



Representation as Term

$cons(a,cons(b,cons(c,cons(d,nil))))$

Language

Objconst: a, b, c, d, nil

Funconst: $cons$

Relconst: $member$

Membership axioms:

$member(x,cons(x,y))$

$member(x,cons(y,z)) \Leftarrow member(x,z)$

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Special Cases of Relational Logic

Ground Logic

no variables, no functions, no quantifiers

Universal Logic

no functions, no quantifiers

free variables implicitly universally quantified

Existential Logic

no functions

Functional Logic

no quantifiers

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Limitations of Ground Logic

Everybody loves somebody.

loves(joe, mary) or loves(joe, sally) or ...

loves(mary, joe) or loves(mary, bill) or ...

The sum of two natural numbers is greater than either number.

$$1+1>1$$

$$1+2>1$$

$$1+2>2$$

...

What about facts about the real numbers?

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Limitations of Universal Logic

Every number is smaller than some number.

Universal Logic

$$\begin{aligned}x < y & \text{ no!} \\ x < a & \text{ no!}\end{aligned}$$

Existential Logic

$$\forall x. \exists y. x < y$$

Functional Logic

$$x < f(x)$$

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Existential and Universal Quantifiers

$\exists x.p(x)$ is *true*.

iff, for some x , $p(x)$ is *true*.

iff it is not *true* that, for all x , $p(x)$ is *false*.

iff it is not *true* that, for all x , $\neg p(x)$ is *true*.

iff it is not *true* that $\forall x. \neg p(x)$ is *true*.

iff $\neg \forall x. \neg p(x)$ is *true*.

In general, $\exists v. \varphi$ is equivalent to $\neg \forall v. \neg \varphi$.

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Need for Explicit Quantifiers

Since $\exists v.\varphi(v)$ is equivalent to $\neg\forall v.\neg\varphi(v)$ and $\varphi(v)$ in universal logic is equivalent to $\forall v.\varphi(v)$, can we express existentials in Universal Logic by negating?

How do we say that somebody loves Mike?

$loves(x,mike)$

This says that everyone loves Mike. Intended:

$\exists x.loves(x,mike)$
 $\neg\forall x.\neg loves(x,mike)$
 $\neg\neg loves(x,mike)$ No!
 $loves(x,mike)$

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Existential Quantifiers and Functions

Functions Replaced by Existential Quantifiers:

$loves(x,a(x)) \Leftrightarrow \exists y.loves(x,y)$

Existential Quantifiers Replaced by Functions:

$\exists y.loves(x,y) \Leftrightarrow loves(x,a(x))$

Theorem: An existential sentence is satisfiable iff the corresponding functional sentence is satisfiable.

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