Computational Logic Lecture 3

# Truth Table Method and Propositional Proofs

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## Deduction

In deduction, the conclusion is true whenever the premises are true.

Premise: $p$
Conclusion: $(p \vee q)$

Premise: $p$
Non-Conclusion: $(p \wedge q)$

Premises: $p, q$
Conclusion: $(p \wedge q)$

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## Logical Entailment

A set of premises Δ *logically entails* a conclusion φ (written as Δ |= φ) if and only if every interpretation that satisfies the premises also satisfies the conclusion.

$\{p\} \models (p \vee q)$

$\{p\}$ |# $(p \wedge q)$

$\{p, q\} \models (p \wedge q)$

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## Logical Entailment ≠ Logical Equivalence

$\{p\} \models (p \vee q)$

$\{p \vee q)\}$ |# $p$

*Analogy in arithmetic: inequalities rather than equations*

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## Truth Table Method

We can check for logical entailment by comparing tables of all possible interpretations.

In the first table, eliminate all rows that do not satisfy premises.

In the second table, eliminate all rows that do not satisfy the conclusion.

If the remaining rows in the first table are a subset of the remaining rows in the second table, then the premises logically entail the conclusion.

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## Example

Does $p$ logically entail $(p \vee q)$?

| $p$ | $q$ |
|---|---|
| T | T |
| T | F |
| F | T |
| F | F |

| $p$ | $q$ |
|---|---|
| T | T |
| T | F |
| F | T |
| F | F |

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# Example

Does $p$ logically entail $(p \wedge q)$?

| $p$ | $q$ |
|---|---|
| T | T |
| T | F |
| F | T |
| F | F |

| $p$ | $q$ |
|---|---|
| T | T |
| T | F |
| F | T |
| F | F |

Does $\{p,q\}$ logically entail $(p \wedge q)$?

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# Example

If Mary loves Pat, then Mary loves Quincy.
If it is Monday, then Mary loves Pat or Quincy.
If it is Monday, does Mary love Quincy?

| $m$ | $p$ | $q$ |
|---|---|---|
| T | T | T |
| × | × | × |
| T | F | T |
| × | × | × |
| F | T | T |
| × | × | × |
| F | F | T |
| F | F | F |

| $m$ | $p$ | $q$ |
|---|---|---|
| T | T | T |
| × | × | × |
| T | F | T |
| × | × | × |
| F | T | T |
| F | T | F |
| F | F | T |
| F | F | F |

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## Logical Entailment and Satisfiability

Theorem: $\Delta \models \varphi$ if and only if $\Delta \cup \{\neg\varphi\}$ is unsatisfiable.

Suppose that $\Delta \models \varphi$. If an interpretation satisfies $\Delta$, then it must also satisfy $\varphi$. But then it cannot satisfy $\neg\varphi$. Therefore, $\Delta \cup \{\neg\varphi\}$ is unsatisfiable.

Suppose that $\Delta \cup \{\neg\varphi\}$ is unsatisfiable. Then every interpretation that satisfies $\Delta$ must *fail* to satisfy $\neg\varphi$, i.e. it must satisfy $\varphi$. Therefore, $\Delta \models \varphi$.

Upshot: We can determine logical entailment by determining unsatisfiability.

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## Example

Problem: $\{(p \Rightarrow q), (m \Rightarrow p \vee q)\} \models (m \Rightarrow q)$?
Or: Is $\{(p \Rightarrow q), (m \Rightarrow p \vee q), \neg(m \Rightarrow q)\}$ unsatisfiable?

| *m* | *p* | *q* |
|---|---|---|
| T | T | T |
| T | T | F |
| T | F | T |
| T | F | F |
| F | T | T |
| F | T | F |
| F | F | T |
| F | F | F |

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## Problem

There can be many, many interpretations for a Propositional Language.

Remember that, for a language with $n$ constants, there are $2^n$ possible interpretations.

Sometimes there are many constants among premises that are irrelevant to the conclusion. Much wasted work.

Answer: Proofs

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## Patterns

A *pattern* is a parameterized expression, i.e. an expression satisfying the grammatical rules of our language except for the use of meta-variables (Greek letters) in place of various subparts of the expression.

Sample Pattern:

$$\varphi \Rightarrow (\psi \Rightarrow \varphi)$$

Instance:

$$p \Rightarrow (q \Rightarrow p)$$

Instance:

$$(p \Rightarrow r) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$$

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# Rules of Inference

A *rule of inference* is a rule of reasoning consisting of one set of sentence patterns, called *premises*, and a second set of sentence patterns, called *conclusions*.

$$\frac{\begin{array}{l}\varphi \Rightarrow \psi \\ \varphi\end{array}}{\psi}$$

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# Rule Instances

An *instance* of a rule of inference is a rule in which all meta-variables have been consistently replaced by expressions in such a way that all premises and conclusions are syntactically legal sentences.

$$\frac{\begin{array}{l}raining \Rightarrow wet \\ raining\end{array}}{wet} \qquad \frac{\begin{array}{l}wet \Rightarrow slippery \\ wet\end{array}}{slippery}$$

$$\frac{\begin{array}{l}p \Rightarrow (q \Rightarrow r) \\ p\end{array}}{q \Rightarrow r} \qquad \frac{\begin{array}{l}(p \Rightarrow q) \Rightarrow r \\ p \Rightarrow q\end{array}}{r}$$

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# Sound Rules of Inference

A rule of inference is *sound* if and only if the premises in any instance of the rule logically entail the conclusions.

Modus Ponens (MP)

$$\frac{\varphi \Rightarrow \psi \qquad \varphi}{\psi}$$

Modus Tolens (MT)

$$\frac{\varphi \Rightarrow \psi \qquad \neg\psi}{\neg\varphi}$$

Equivalence Elimination (EE)

$$\frac{\varphi \Leftrightarrow \psi}{\varphi \Rightarrow \psi \qquad \psi \Rightarrow \varphi}$$

Double Negation (DN)

$$\frac{\neg\neg\varphi}{\varphi}$$



# Proof (Version 1)

A *proof* of a conclusion from a set of premises is a sequence of sentences terminating in the conclusion in which each item is either:

1. a premise
2. the result of applying a rule of inference to earlier items in sequence.

## Example

When it is raining, the ground is wet. When the ground is wet, it is slippery. It is raining. Prove that it is slippery.

| | | |
|---|---|---|
| 1. | $raining \Rightarrow wet$ | Premise |
| 2. | $wet \Rightarrow slippery$ | Premise |
| 3. | $raining$ | Premise |
| 4. | $wet$ | MP : 1, 3 |
| 5. | $slippery$ | MP : 2, 4 |

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## Error

Note: Rules of inference apply only to top-level sentences in a proof. Sometimes works but sometimes fails.

| | | | | |
|---|---|---|---|---|
| | 1. | $raining \Rightarrow cloudy$ | Premise | |
| | 2. | $raining \Rightarrow wet$ | Premise | |
| No! | 3. | $cloudy \Rightarrow wet$ | MP : 1, 2 | No! |

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# Example

Heads you win. Tails I lose. Suppose the coin comes up tails. Show that you win.

| | | |
|---|---|---|
| 1. | $h \Rightarrow y$ | Premise |
| 2. | $t \Rightarrow \neg m$ | Premise |
| 3. | $h \Leftrightarrow \neg t$ | Premise |
| 4. | $y \Leftrightarrow \neg m$ | Premise |
| 5. | $t$ | Premise |
| 6. | $\neg m$ | MP : 2, 5 |
| 7. | $y \Rightarrow \neg m$ | EE : 4 |
| 8. | $\neg m \Rightarrow y$ | EE : 4 |
| 9. | $y$ | MP : 8, 6 |

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# Axiom Schemata

Fact: If a sentence is valid, then it is true under all interpretations. Consequently, there should be a proof without making any assumptions at all.

Fact: $(p \Rightarrow (q \Rightarrow p))$ is a valid sentence.
Problem: Prove $(p \Rightarrow (q \Rightarrow p))$.

Solution: We need some rules of inference without premises to get started.

An *axiom schema* is sentence pattern construed as a rule of inference without premises.

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## Rules and Schemata

Axiom Schemata as Rules of Inference

$$\varphi \Rightarrow (\psi \Rightarrow \varphi) \qquad \frac{}{\varphi \Rightarrow (\psi \Rightarrow \varphi)}$$

Rules of Inference as Axiom Schemata

$$\frac{\begin{array}{l}\varphi \Rightarrow \psi \\ \neg\psi\end{array}}{\neg\varphi} \qquad (\varphi \Rightarrow \psi) \Rightarrow (\neg\psi \Rightarrow \neg\varphi)$$

Note: Of course, we must keep a least one rule of inference to use the schemata. By convention, we retain Modus Ponens.

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## Valid Axiom Schemata

A *valid axiom schema* is a sentence pattern denoting an infinite set of sentences, all of which are valid.

Implication Introduction (II):

$$\varphi \Rightarrow (\psi \Rightarrow \varphi)$$

ImplicationDistribution (ID):

$$(\varphi \Rightarrow (\psi \Rightarrow \chi)) \Rightarrow ((\varphi \Rightarrow \psi) \Rightarrow (\varphi \Rightarrow \chi))$$

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## Proof (Official Version)

A *proof* of a conclusion from a set of premises is a sequence of sentences terminating in the conclusion in which each item is either:

1. a premise
2. An instance of an axiom schema
3. the result of applying a rule of inference to earlier items in sequence.

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## Sample Proof

Whenever $p$ is true, $q$ is true. Whenever $q$ is true, $r$ is true. Prove that, whenever $p$ is true, $r$ is true.

| | | |
|---|---|---|
| 1. | $p \Rightarrow q$ | Premise |
| 2. | $q \Rightarrow r$ | Premise |
| 3. | $(q \Rightarrow r) \Rightarrow (p \Rightarrow (q \Rightarrow r))$ | II |
| 4. | $p \Rightarrow (q \Rightarrow r)$ | MP : 3, 2 |
| 5. | $(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$ | ID |
| 6. | $(p \Rightarrow q) \Rightarrow (p \Rightarrow r)$ | MP : 5, 4 |
| 7. | $p \Rightarrow r$ | MP : 6, 1 |

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## Mendelson Axiomatization

II: $\varphi \Rightarrow (\psi \Rightarrow \varphi)$
ID: $(\varphi \Rightarrow (\psi \Rightarrow \chi)) \Rightarrow ((\varphi \Rightarrow \psi) \Rightarrow (\varphi \Rightarrow \chi))$
CR: $(\neg\psi \Rightarrow \varphi) \Rightarrow ((\neg\psi \Rightarrow \neg\varphi) \Rightarrow \psi)$

Note: Mendelson's system assumes there are only two operators, viz. $\neg$ and $\Rightarrow$. Fortunately, all sentences in Propositional Logic can be reduced to equivalent sentences with these operators by applying the following rules.

$(\psi \Leftrightarrow \varphi) \rightarrow ((\varphi \Rightarrow \psi) \wedge (\psi \Rightarrow \varphi))$
$(\varphi \Leftarrow \psi) \rightarrow (\psi \Rightarrow \varphi)$
$(\psi \wedge \varphi) \rightarrow \neg(\neg\varphi \Rightarrow \psi)$
$(\psi \vee \varphi) \rightarrow (\neg\varphi \Rightarrow \psi)$

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## Kleene Axiomatization

II: $\varphi \Rightarrow (\psi \Rightarrow \varphi)$
ID: $(\varphi \Rightarrow (\psi \Rightarrow \chi)) \Rightarrow ((\varphi \Rightarrow \psi) \Rightarrow (\varphi \Rightarrow \chi))$

AI: $\varphi \Rightarrow (\psi \Rightarrow (\varphi \wedge \psi))$
AE1: $(\varphi \wedge \psi) \Rightarrow \varphi$
AE2: $(\varphi \wedge \psi) \Rightarrow \psi$

OI1: $\varphi \Rightarrow (\varphi \vee \psi)$
OI2: $\psi \Rightarrow (\varphi \vee \psi)$
OE: $(\varphi \Rightarrow \chi) \Rightarrow ((\psi \Rightarrow \chi) \Rightarrow (\varphi \vee \psi \Rightarrow \chi))$

CM: $(\psi \Rightarrow \varphi) \Rightarrow ((\psi \Rightarrow \neg\varphi) \Rightarrow \neg\psi)$
DN: $(\neg\neg\varphi \Rightarrow \varphi)$

Note: Kleene's system assumes there are only four operators, viz. $\wedge$, $\vee$, $\neg$, and $\Rightarrow$.

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## Standard Axiom Schemata

II: $\varphi \Rightarrow (\psi \Rightarrow \varphi)$
ID: $(\varphi \Rightarrow (\psi \Rightarrow \chi)) \Rightarrow ((\varphi \Rightarrow \psi) \Rightarrow (\varphi \Rightarrow \chi))$
CR: $(\neg\psi \Rightarrow \varphi) \Rightarrow ((\neg\psi \Rightarrow \neg\varphi) \Rightarrow \psi)$

EQ: $(\varphi \Leftrightarrow \psi) \Rightarrow (\varphi \Rightarrow \psi)$
$(\varphi \Leftrightarrow \psi) \Rightarrow (\psi \Rightarrow \varphi)$
$(\varphi \Rightarrow \psi) \Rightarrow ((\psi \Rightarrow \varphi) \Rightarrow (\varphi \Leftrightarrow \psi))$

OQ: $(\varphi \Leftarrow \psi) \Leftrightarrow (\psi \Rightarrow \varphi)$
$(\varphi \vee \psi) \Leftrightarrow (\neg\varphi \Rightarrow \psi)$
$(\varphi \wedge \psi) \Leftrightarrow \neg(\neg\varphi \vee \neg\psi)$

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## Meredith Axiomatization

$$((((\varphi \Rightarrow \psi) \Rightarrow (\neg\chi \Rightarrow \neg\mu)) \Rightarrow \chi) \Rightarrow \nu)$$
$$\Rightarrow$$
$$((\nu \Rightarrow \varphi) \Rightarrow (\mu \Rightarrow \varphi))$$

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## Provability

A conclusion is said to be *provable* from a set of premises (written $\Delta \vdash \varphi$) if and only if there is a finite proof of the conclusion from the premises using only Modus Ponens and a complete logical axiomatization (e.g. Mendelson, Kleene, Standard, Meredith).

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## Soundness and Completeness

Soundness: Our proof system is *sound*, i.e. if the conclusion is provable from the premises, then the premises propositionally entail the conclusion.

$$(\Delta \vdash \varphi) \Rightarrow (\Delta \models \varphi)$$

Completeness: Our proof system is *complete*, i.e. if the premises propositionally entail the conclusion, then the conclusion is provable from the premises.

$$(\Delta \models \varphi) \Rightarrow (\Delta \vdash \varphi)$$

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# Truth Tables and Proofs

The truth table method and the proof method succeed in exactly the same cases.

On large problems, the proof method often takes fewer steps than the truth table method. However, in the worst case, the proof method may take just as many or more steps to find an answer as the truth table method.

Usually, proofs are much smaller than the corresponding truth tables. So writing an argument to convince others does not take as much space.

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# Metatheorems

Deduction Theorem: $\Delta \vdash (\varphi \Rightarrow \psi)$ if and only if $\Delta \cup \{\varphi\} \vdash \psi$.

Substitution Theorem: $\Delta \vdash (\varphi \Leftrightarrow \psi)$ and $\Delta \vdash \chi$, then it is the case that $\Delta \vdash \chi_{\varphi \leftarrow \psi}$.

Chaining Theorem: If $\Delta \vdash (\varphi \Rightarrow \psi)$ and $\Delta \vdash (\psi \Rightarrow \chi)$, then $\Delta \vdash (\varphi \Rightarrow \chi)$.

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## Proof Without Metatheorems

Problem: $\{p \Rightarrow q, q \Rightarrow r\} \vdash (p \Rightarrow r)$?

| | | |
|---|---|---|
| 1. | $p \Rightarrow q$ | Premise |
| 2. | $q \Rightarrow r$ | Premise |
| 3. | $(q \Rightarrow r) \Rightarrow (p \Rightarrow (q \Rightarrow r))$ | II |
| 4. | $p \Rightarrow (q \Rightarrow r)$ | MP : 3, 2 |
| 5. | $(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$ | ID |
| 6. | $(p \Rightarrow q) \Rightarrow (p \Rightarrow r)$ | MP : 5, 4 |
| 7. | $p \Rightarrow r$ | MP : 6, 1 |

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## Proof Using Deduction Theorem

Problem: $\{p \Rightarrow q, q \Rightarrow r\} \vdash (p \Rightarrow r)$?

| | | |
|---|---|---|
| 1. | $p \Rightarrow q$ | Premise |
| 2. | $q \Rightarrow r$ | Premise |
| 3. | $p$ | Premise |
| 4. | $q$ | MP : 1, 3 |
| 5. | $r$ | MP : 2, 4 |

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# TA Appeasement Rules

When we ask you to *show* that something is true, you may use metatheorems.

When we ask you to give a *formal proof*, it means you should write out the proof as defined above.

When we ask you to give a formal proof *using* certain rules of inference or axiom schemata, it means you should do so using *only* those rules of inference and axiom schemata and no others.

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