Computational Logic Lecture 2

Propositional Logic

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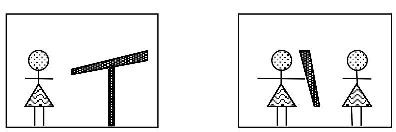
Ambiguity

There’s a girl in the room with a telescope.



Compound Sentences

Negations:

$\neg$*raining*

The argument of a negation is called the *target*.

Conjunctions:

(*raining* ∧ *snowing*)

The arguments of a conjunction are called *conjuncts*.

Disjunctions:

(*raining* ∨ *snowing*)

The arguments of a disjunction are called *disjuncts*.

Compound Sentences (concluded)

Implications:

(*raining* ⇒ *cloudy*)

The left argument of an implication is the *antecedent*.
The right argument is the *consequent*.

Reductions:

(*cloudy* ⇐ *raining*)

The left argument of a reduction is the *consequent*.
The right argument of a reduction is the *antecedent*.

Equivalences:

(*cloudy* ⇔ *raining*)

Parenthesis Removal

Dropping Parentheses is good:

$$(p \land q) \rightarrow p \land q$$

But it can lead to ambiguities:

$$((p \lor q) \land r) \rightarrow p \land q \lor r$$

$$(p \lor (q \land r)) \rightarrow p \land q \lor r$$

Precedence

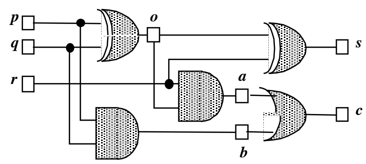
Parentheses can be dropped when the structure of an expression can be determined by precedence.

$$\neg$$
$$\land$$
$$\lor$$
$$\Rightarrow \Leftarrow \Leftrightarrow$$

NB: An operand associates with operator of higher precedence. If surrounded by operators of equal precedence, the operand associates with the operator to the right.

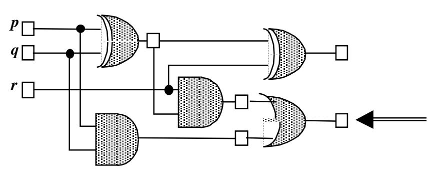
$p \land q \lor r$	$p \Rightarrow q \Rightarrow r$	$\neg p \land q$
$p \lor q \land r$	$p \Rightarrow q \Leftarrow r$	

Example



$o \Leftrightarrow (p \wedge \neg q) \vee (\neg p \wedge q)$
$a \Leftrightarrow r \wedge o$
$b \Leftrightarrow p \wedge q$
$s \Leftrightarrow (o \wedge \neg r) \vee (\neg o \wedge r)$
$c \Leftrightarrow a \vee b$

Example



$(r \wedge ((p \wedge \neg q) \vee (\neg p \wedge q))) \vee (p \wedge q)$

Propositional Interpretation

A *propositional interpretation* is an association between the propositional constants in a propositional language and the truth values T or F.

$p \xrightarrow{i} T$	$p^i = T$
$q \xrightarrow{i} F$	$q^i = F$
$r \xrightarrow{i} T$	$r^i = T$

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Sentential Interpretation

A *sentential interpretation* is an association between the sentences in a propositional language and the truth values T or F.

$p^i = T$	$(p \vee q)^i = T$
$q^i = F$	$(\neg q \vee r)^i = T$
$r^i = T$	$((p \vee q) \wedge (\neg q \vee r))^i = T$

A propositional interpretation defines a sentential interpretation by application of operator semantics.

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Operator Semantics

Negation:

ϕ	$\neg\phi$
T	F
F	T

For example, if the interpretation of p is F, then the interpretation of $\neg p$ is T.

For example, if the interpretation of $(p \wedge q)$ is T, then the interpretation of $\neg(p \wedge q)$ is F.

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Operator Semantics (continued)

Conjunction:

ϕ	ψ	$\phi \wedge \psi$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction:

ϕ	ψ	$\phi \vee \psi$
T	T	T
T	F	T
F	T	T
F	F	F

NB: The type of disjunction here is called *inclusive or*, which says that a disjunction is true if and only if *at least* one of its disjuncts is true. This contrasts with *exclusive or*, which says that a disjunction is true if and only if an odd number of its disjuncts is true.

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Operator Semantics (continued)

Implication:

ϕ	ψ	$\phi \Rightarrow \psi$
T	T	T
T	F	F
F	T	T
F	F	T

Reduction:

ϕ	ψ	$\phi \Leftarrow \psi$
T	T	T
T	F	T
F	T	F
F	F	T

NB: The semantics of implication here is called *material implication*. Any implication is true if the antecedent is false, whether or not there is a connection to the consequent.

If George Washington is alive, I am a billionaire.

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Operator Semantics (concluded)

Equivalence:

ϕ	ψ	$\phi \Leftrightarrow \psi$
T	T	T
T	F	F
F	T	F
F	F	T

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Evaluation

Interpretation *i*:

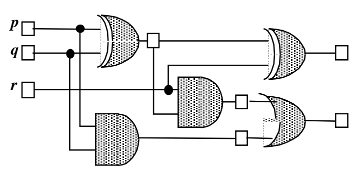
$$p^i = \text{T}$$
$$q^i = \text{F}$$
$$r^i = \text{T}$$

Compound Sentence

$$(p \vee q) \wedge (\neg q \vee r)$$

Example

$p^i = \text{T}$
$q^i = \text{T}$
$r^i = \text{T}$



$$(r \wedge ((p \wedge \neg q) \vee (\neg p \wedge q))) \vee (p \wedge q)$$

Multiple Interpretations

Logic does not prescribe which interpretation is "correct". In the absence of additional information, one interpretation is as good as another.

Interpretation *i*

p^i = T
q^i = F
r^i = T

Interpretation *j*

p^j = F
q^j = F
r^j = T

Examples:

- Different days of the week
- Different locations
- Beliefs of different people

Truth Tables

A *truth table* is a table of all possible interpretations for the propositional constants in a language.

p	*q*	*r*
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

One column per constant.

One row per interpretation.

For a language with *n* constants, there are 2^n interpretations.

Properties of Sentences

Valid

Contingent

Unsatisfiable

A sentence is *valid* if and only if *every* interpretation satisfies it.

A sentence is *contingent* if and only if *some* interpretation satisfies it and *some* interpretation falsifies it.

A sentence is *unsatisfiable* if and only if *no* interpretation satisfies it.

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Properties of Sentences

Valid

Contingent

Unsatisfiable

} A sentences is *satisfiable* if and only if it is either valid or contingent.

} A sentences is *falsifiable* if and only if it is contingent or unsatisfiable.

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Example of Validity

p	q	r	$(p \Rightarrow q)$	$(q \Rightarrow r)$	$(p \Rightarrow q) \vee (q \Rightarrow r)$
1	1	1			
1	1	0			
1	0	1			
1	0	0			
0	1	1			
0	1	0			
0	0	1			
0	0	0			

More Validities

Double Negation:

$$p \Leftrightarrow \neg\neg p$$

deMorgan's Laws:

$$\neg(p \wedge q) \Leftrightarrow (\neg p \vee \neg q)$$
$$\neg(p \vee q) \Leftrightarrow (\neg p \wedge \neg q)$$

Implication Introduction:

$$p \Rightarrow (q \Rightarrow p)$$

Implication Distribution

$$(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$$

Evaluation Versus Satisfaction

Evaluation:

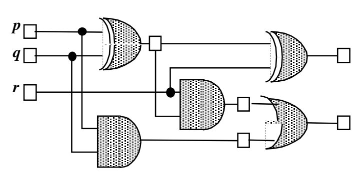
$$p^i = \mathrm{T} \qquad q^i = \mathrm{F} \quad \longrightarrow \quad (p \vee q)^i = \mathrm{T} \qquad (\neg q)^i = \mathrm{T}$$

Satisfaction:

$$(p \vee q)^i = \mathrm{T} \qquad (\neg q)^i = \mathrm{T} \quad \longrightarrow \quad p^i = \mathrm{T} \qquad q^i = \mathrm{F}$$

Example

$p^i = ?$
$q^i = ?$
$r^i = ?$



$$((r \wedge ((p \wedge \neg q) \vee (\neg p \wedge q))) \vee (p \wedge q))^i = \mathrm{T}$$

Satisfaction

Method to find all propositional interpretations that satisfy a given set of sentences:

(1) Form a truth table for the propositional constants.

(2) For each sentence in the set and each row in the truth table, check whether the row satisfies the sentence. If not, cross out the row.

(3) Any row remaining satisfies all sentences in the set. (Note that there might be more than one.)

Satisfaction Example

$q \Rightarrow r$

p	q	r	
T	T	T	
T	T	F	×
T	F	T	
T	F	F	
F	T	T	
F	T	F	×
F	F	T	
F	F	F	

Satisfaction Example (continued)

$q \Rightarrow r$

$p \Rightarrow q \wedge r$

p	q	r	
T	T	T	
T	T	F	×
T	F	T	×
T	F	F	×
F	T	T	
F	T	F	×
F	F	T	
F	F	F	

Satisfaction Example (concluded)

$q \Rightarrow r$

$p \Rightarrow q \wedge r$

$\neg r$

p	q	r	
T	T	T	×
T	T	F	×
T	F	T	×
T	F	F	×
F	T	T	×
F	T	F	×
F	F	T	×
F	F	F	

The Big Game

Stanford people always tell the truth, and Berkeley people always lie. Unfortunately, by looking at a person, you cannot tell whether he is from Stanford or Berkeley.

You come to a fork in the road and want to get to the football stadium down one fork. However, you do not know which to take. There is a person standing there. What single question can you ask him to help you decide which fork to take?

Basic Idea

left	*su*	*Question*	*Response*
T	T		
T	F		
F	T		
F	F		

The Big Game Solved

Question: *The left road the way to the stadium if and only if you are from Stanford. Is that correct?*

$$left \Leftrightarrow su$$