

Mathematical Induction

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Incomplete Induction

Definition

$$f(1)=1$$
$$f(x+1)=f(x)+2*x+1$$

Data

$$1 = 1 = 1^2$$
$$1+3 = 4 = 2^2$$
$$1+3+5 = 9 = 3^2$$
$$1+3+5+7 = 16 = 4^2$$
$$1+3+5+7+9 = 25 = 5^2$$

Conjecture

$$f(x) = x^2$$

In this case, the answer is correct. Lucky Guess.

Not So Lucky Guess

Data:

$$2^{2^1} + 1 = 2^{2+1} = 5$$

$$2^{2^2} + 1 = 2^{4+1} = 17$$

$$2^{2^3} + 1 = 2^{8+1} = 257$$

$$2^{2^4} + 1 = 2^{16+1} = 65537$$

“Theorem” by Fermat (1601-1665):

$$\text{prime}(2^{2^x} + 1)$$

Fact discovered (mercifully) after his death:

$$2^{2^5} + 1 = 4,294,967,297 = 641 * 6,700,417$$

Oops.

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Basic Metavocabulary

Data

$$1 = 1 = 1^2$$

$$1+3 = 4 = 2^2$$

$$1+3+5 = 9 = 3^2$$

$$1+3+5+7 = 16 = 4^2$$

Base Case: Prove for $n=1$

$$f(1)=1=1^2$$

Inductive Case: Assume true for x ; prove for $x+1$

$$f(x+1)=f(x)+2*x+1$$

$$f(x+1)=x^2+2*x+1$$

$$f(x+1)=(x+1)^2$$

Jules Henri Poincare (1854-1912) credited with invention.

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Outline

Linear Induction

Input: successor function on individuals

Output: universally quantified conclusions

Structural Induction

Input: constructor function for structures

Output: universally quantified conclusions

Ordered Induction

Input: ordering relation on individuals

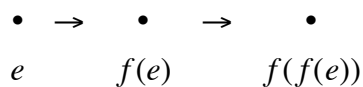
Output: universally quantified conclusions

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Linear Induction

Linearly Structured World:



In other words, there is a distinguished *base element* and there is a *successor function*, which, starting at the base element, *enumerates all elements* in the universe of discourse.

Base element: e

Successor function: f

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Linear Induction Schema

If a property holds of the base element and if it holds of a successor whenever it holds of an element, we would like to assert that it holds of all elements in the universe of discourse.

$$\phi[e] \wedge \forall x.(\phi[x] \Rightarrow \phi[f(x)]) \Rightarrow \forall x.\phi[x]$$

Base case: $\phi[e]$

Inductive case: $\forall x.(\phi[x] \Rightarrow \phi[f(x)])$

Inductive antecedent: $\phi[x]$

Inductive consequent: $\phi[f(x)]$

Conclusion: $\forall x.\phi[x]$

Analogy to students in class passing on messages

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Arithmetic Examples

Object constant: 0

Unary function constant: s (+1)

Unary relation constants: p , $even$, odd

Induction Schema:

$$\phi[0] \wedge \forall x.(\phi[x] \Rightarrow \phi[s(x)]) \Rightarrow \forall x.\phi[x]$$

Instances of Induction Schema:

$$p(0) \wedge \forall x.(p(x) \Rightarrow p(s(x))) \Rightarrow \forall x.p(x)$$

$$even(0) \wedge \forall x.(even(x) \Rightarrow even(s(x))) \Rightarrow \forall x.even(x)$$

$$(even(0) \vee odd(0))$$

$$\wedge \forall x.(even(x) \vee odd(x) \Rightarrow even(s(x)) \vee odd(s(x)))$$

$$\Rightarrow \forall x.(even(x) \vee odd(x))$$

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Arithmetic Problem

Object constant: 0
 Unary function constant: s (+1)
 Unary relation constants: p

Axioms

$$p(0) \\ \forall x.(p(x) \Rightarrow p(s(x)))$$

Goal:

$$\forall x.p(x)$$

Induction Schema:

$$p(0) \wedge \forall x.(p(x) \Rightarrow p(s(x))) \Rightarrow \forall x.p(x)$$

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Clausal Form

$$p(0) \wedge \forall x.(p(x) \Rightarrow p(s(x))) \Rightarrow \forall x.p(x)$$

I: $\neg(p(0) \wedge \forall x.(\neg p(x) \vee p(s(x)))) \vee \forall x.p(x)$
 N: $\neg p(0) \vee \neg \forall x.(\neg p(x) \vee p(s(x))) \vee \forall x.p(x)$
 $\neg p(0) \vee \exists x. \neg(\neg p(x) \vee p(s(x))) \vee \forall x.p(x)$
 $\neg p(0) \vee \exists x.(\neg \neg p(x) \wedge \neg p(s(x))) \vee \forall x.p(x)$
 $\neg p(0) \vee \exists x.(p(x) \wedge \neg p(s(x))) \vee \forall x.p(x)$

S:
 E: $\neg p(0) \vee (p(a) \wedge \neg p(s(a))) \vee \forall x.p(x)$
 A: $\neg p(0) \vee (p(a) \wedge \neg p(s(a))) \vee p(x)$
 D: $\neg p(0) \vee p(a) \vee p(x)$
 $\neg p(0) \vee \neg p(s(a)) \vee p(x)$

O: $\{\neg p(0), p(a), p(x)\}$
 $\{\neg p(0), \neg p(s(a)), p(x)\}$

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Resolution Proof

1.	$\{p(0)\}$	Premise
2.	$\{\neg p(x), p(s(x))\}$	Premise
3.	$\{\neg p(0), p(a), p(x)\}$	Induction
4.	$\{\neg p(0), \neg p(s(a)), p(x)\}$	Induction
5.	$\{\neg p(c)\}$	Goal
6.	$\{p(a), p(x)\}$	1, 3
7.	$\{\neg p(s(a)), p(x)\}$	1, 4
8.	$\{p(s(a)), p(x)\}$	2, 6
9.	$\{p(x)\}$	7, 8
10.	$\{\}$	5, 9

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Arithmetic Problem

Object constant: 0
 Unary function constant: s
 Binary relation constants: $even, odd$

Axioms

$$\begin{aligned}
 & even(0) \\
 & \forall x.(even(x) \Rightarrow odd(s(x))) \\
 & \forall x.(odd(x) \Rightarrow even(s(x)))
 \end{aligned}$$

Goal:

$$\forall x.(even(x) \vee odd(x))$$

Induction Schema:

$$\begin{aligned}
 & (even(0) \vee odd(0)) \\
 & \wedge \forall x.(even(x) \vee odd(x) \Rightarrow even(s(x)) \vee odd(s(x))) \\
 & \Rightarrow \forall x.(even(x) \vee odd(x))
 \end{aligned}$$

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Clausal Form

$$\begin{aligned} & (even(0) \vee odd(0)) \\ & \wedge \forall x. (even(x) \vee odd(x) \Rightarrow even(s(x)) \vee odd(s(x))) \\ & \Rightarrow \forall x. (even(x) \vee odd(x)) \end{aligned}$$
$$\begin{aligned} & \{\neg even(0), even(a), odd(a), even(x), odd(x)\} \\ & \{\neg odd(0), even(a), odd(a), even(x), odd(x)\} \\ & \{\neg even(0), \neg even(s(a)), even(x), odd(x)\} \\ & \{\neg even(0), \neg odd(s(a)), even(x), odd(x)\} \\ & \{\neg odd(0), \neg even(s(a)), even(x), odd(x)\} \\ & \{\neg odd(0), \neg odd(s(a)), even(x), odd(x)\} \end{aligned}$$

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Induction and Resolution

Understandability

Okay for computers
Terrible for humans

Generality:

Induction is an axiom schema.
Resolution works on sentences.

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Linear Induction Method

Using the Induction schema to prove a universally quantified formula.

(1) Base Case. Prove the base case.

(2) Inductive Case. Assume ground version of induction antecedent (induction hypothesis) and prove corresponding version of induction consequent.

If successful, the universally quantified conclusion holds. Why? Deduction Theorem.

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Even and Odd

Axioms

$$\begin{aligned} & \text{even}(0) \\ & \text{even}(x) \Rightarrow \text{odd}(s(x)) \\ & \text{odd}(x) \Rightarrow \text{even}(s(x)) \end{aligned}$$

Desired Conclusion:

$$\forall x. (\text{even}(x) \vee \text{odd}(x))$$

Induction Axiom:

$$\begin{aligned} & (\text{even}(0) \vee \text{odd}(0)) \\ & \wedge \forall x. (\text{even}(x) \vee \text{odd}(x) \Rightarrow \text{even}(s(x)) \vee \text{odd}(s(x))) \\ & \Rightarrow \forall x. (\text{even}(x) \vee \text{odd}(x)) \end{aligned}$$

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Even and Odd Problem

- | | | |
|-----|-------------------------------|-----------|
| 1. | $\{even(0)\}$ | Induction |
| 2. | $\{\neg even(x), odd(s(x))\}$ | Premise |
| 3. | $\{\neg odd(x), even(s(x))\}$ | Axiom |
| 4. | $\{even(a), odd(a)\}$ | Induction |
| 5. | $\{\neg even(s(a))\}$ | Goal |
| 6. | $\{\neg odd(s(a))\}$ | Goal |
| 7. | $\{\neg odd(a)\}$ | 2, 6 |
| 8. | $\{\neg even(a)\}$ | 3, 5 |
| 9. | $\{even(a)\}$ | 4, 7 |
| 10. | $\{\}$ | 8, 9 |

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Binary Addition

Object constant: 0

Unary function constant: s

Binary function constant: $+$

Chain Axioms:

$$s(x)=s(y) \Rightarrow x=y$$

Addition Axioms:

$$x+0=x$$

$$x+s(y)=s(x+y)$$

$$s(x)+y=s(x+y)$$

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Binary Addition Problem

Question:

$$\forall x. 0+x=x$$

Axioms:

$$s(x)=s(y) \Rightarrow x=y$$

$$x+0=x$$

$$x+s(y)=s(x+y)$$

$$s(x)+y=s(x+y)$$

Goal:

$$\forall x. 0+x=x$$

Induction Axiom:

$$0+0=0 \wedge \forall x.(0+x=x \Rightarrow 0+s(x)=s(x)) \Rightarrow \forall x.0+x=x$$

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Binary Addition Problem

1. $\{\neg s(x) = s(y), x = y\}$ Axiom
2. $\{x + 0 = x\}$ Axiom
3. $\{x + s(y) = s(x + y)\}$ Axiom
4. $\{s(x) + y = s(x + y)\}$ Axiom
5. $\{0 + a = a\}$ Induction
6. $\{x = x\}$ Equality
7. $\{\neg 0 + s(a) = s(a)\}$ Goal
8. $\{\neg s(0 + a) = s(a)\}$ 3,7
9. $\{\neg s(a) = s(a)\}$ 5,8
10. $\{\}$ 6,9

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Binary Addition Problem

Axioms:

$$s(x)=s(y) \Rightarrow x=y$$

$$x+0=x$$

$$x+s(y)=s(x+y)$$

$$s(x)+y=s(x+y)$$

Question:

$$\forall x.x+y=y+x$$

Induction Axiom:

$$0+y=y+0 \wedge \forall x.(x+y=y+x \Rightarrow s(x)+y=y+s(x)) \Rightarrow \forall x.x+y=y+x$$

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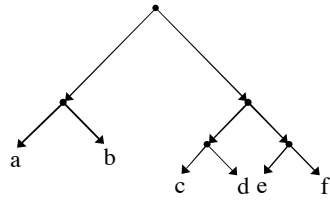
Binary Addition Problem

1. $\{\neg s(x) = s(y), x = y\}$ Axiom
2. $\{x + 0 = x\}$ Axiom
3. $\{x + s(y) = s(x + y)\}$ Axiom
4. $\{s(x) + y = s(x + y)\}$ Axiom
5. $\{a + y = y + a\}$ Induction
6. $\{x = x\}$ Equality
7. $\{\neg s(a) + y = y + s(a)\}$ Goal
8. $\{\neg s(a + y) = y + s(a)\}$ 4,7
9. $\{\neg s(y + a) = y + s(a)\}$ 5,8
10. $\{\neg y + s(a) = y + s(a)\}$ 3,9
11. $\{\}$ 6,10

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Intuition for Structural Induction

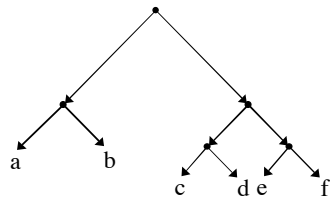


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Binary Tree Representation

Tree:



Representation as a term:

$pair(pair(a,b),pair(pair(c,d),pair(e,f)))$

Infix version:

$(a*b)*((c*d)*(e*f))$

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Induction and Resolution

Base elements: a, b

Successor function: $*$

Induction Schema:

$$\phi[a] \wedge \phi[b] \wedge \forall x. \forall y. (\phi[x] \wedge \phi[y] \Rightarrow \phi[x*y]) \Rightarrow \forall x. \phi[x]$$

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Reverse

Object Constants: a, b

Unary Function Constant: rev

Binary Function Constant: $*$

Axioms:

$$rev(a)=a$$

$$rev(b)=b$$

$$rev(x*y)=rev(y)*rev(x)$$

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Reverse Problem

Desired Conclusion:

$$\text{rev}(\text{rev}(x))=x$$

Axioms:

$$\text{rev}(a)=a$$

$$\text{rev}(b)=b$$

$$\text{rev}(x*y)=\text{rev}(y)*\text{rev}(x)$$

Induction Axiom:

$$\text{rev}(\text{rev}(a))=a \wedge \text{rev}(\text{rev}(b))=b$$

$$\wedge \forall x. \forall y. (\text{rev}(\text{rev}(x))=x \wedge \text{rev}(\text{rev}(y))=y) \Rightarrow \text{rev}(\text{rev}(x*y))=x*y$$

$$\Rightarrow \forall x. \text{rev}(\text{rev}(x))=x$$

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Reverse Problem

- | | |
|---|-----------|
| 1. $\{\text{rev}(a) = a\}$ | Axiom |
| 2. $\{\text{rev}(b) = b\}$ | Axiom |
| 3. $\{\text{rev}(x * y) = \text{rev}(y) * \text{rev}(x)\}$ | Axiom |
| 4. $\{\text{rev}(\text{rev}(c)) = c\}$ | Induction |
| 5. $\{\text{rev}(\text{rev}(d)) = d\}$ | Induction |
| 6. $\{x = x\}$ | Equality |
| 7. $\{\neg \text{rev}(\text{rev}(x * y)) = x * y\}$ | Goal |
| 8. $\{\neg \text{rev}(\text{rev}(y) * \text{rev}(x)) = x * y\}$ | 3,7 |
| 9. $\{\neg \text{rev}(\text{rev}(x)) * \text{rev}(\text{rev}(y)) = x * y\}$ | 3,8 |
| 10. $\{\neg c * \text{rev}(\text{rev}(y)) = c * y\}$ | 4,9 |
| 11. $\{\neg c * d = c * d\}$ | 5,10 |
| 12. $\{\}$ | 6,11 |

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Ordered Induction

Minimal Element: e

Ordering Relation: $<$

Inductive Formula: ϕ

Induction Schema:

$$\forall x. (\forall y. (y < x \Rightarrow \phi[y]) \Rightarrow \phi[x]) \Rightarrow \forall x. \phi[x]$$

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Base Case

Ordered Induction Schema:

$$\forall x. (\forall y. (y < x \Rightarrow \phi[y]) \Rightarrow \phi[x]) \Rightarrow \forall x. \phi[x]$$

It appears that there is no base case. Ah, but there is!
Remember that e is a minimal element.

$$\forall x. \neg x < e$$

Let x be e in induction antecedent.

$$\forall y. (y < e \Rightarrow \phi[y]) \Rightarrow \phi[e]$$

Antecedent of *this* sentence must always be true.
Therefore, to prove the condition, necessary to prove $\phi[e]$.

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Goldbach's Conjecture

Data	6	=	3 + 3
	8	=	3 + 5
	10	=	5 + 5
	12	=	5 + 7
	14	=	7 + 7
	16	=	5 + 11

Conjecture (1742):

$$\forall z. (\text{even}(z) \wedge z > 4 \Rightarrow \exists x. \exists y. (\text{prime}(x) \wedge \text{prime}(y) \wedge x + y = z))$$

*As of late 60's, still not proved.
Single counterexample would disprove.*

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Summary

The key to induction is having some local way of working through space of all objects and using that to establish arbitrary formulas.

Induction Method:

Instance of schema can be used by standard proof methods.

Induction schema cannot be written as finite set of axioms.

Can state as a metalevel axiom and use metareasoning.

Induction method works when schema true, and it's simple.

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