Computational Logic Lecture 2

Propositional Logic

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Grammatical Complexity

The dog *chased* the cat.
The dog that *ate* the rat *chased* the cat.

The cherry *blossoms* in the spring … *sank*.

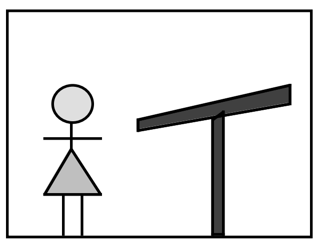
Ambiguity

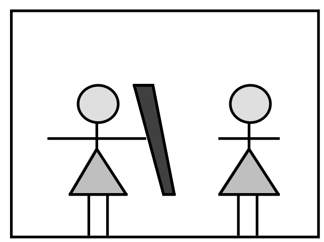
Leland Stanford Junior University

Leland-Stanford   Junior-University?

Leland-Stanford-Junior   University?

There’s a girl in the room with a telescope.





Ambiguity

Lettuce won't turn brown if you put your head
In a plastic bag before placing it in the
refrigerator.

The manager of a nudist colony complains that
a hole was cut in the wall surrounding the camp.
Police are looking into it.

from *Misteaks in Print*
edited by Kermit Shaefer

Propositional Constants

Examples:

raining

*r*32*aining*

rAiNiNg

rainingorsnowing

Non-Examples:

324567

raining.or.snowing

Compound Sentences

Negations:

$$\neg raining$$

The argument of a negation is called the *target*.

Conjunctions:

$$(raining \wedge snowing)$$

The arguments of a conjunction are called *conjuncts*.

Disjunctions:

$$(raining \vee snowing)$$

The arguments of a disjunction are called *conjuncts*.

Compound Sentences (concluded)

Implications:

$$(raining \Rightarrow cloudy)$$

The left argument of an implication is the *antecedent*.
The right argument of an implication is the *consequent*.

Reductions:

$$(cloudy \Leftarrow raining)$$

The left argument of a reduction is the *consequent*.
The right argument of a reduction is the *antecedent*.

Equivalences:

$$(cloudy \Leftrightarrow raining)$$

Parentheses Examples

Dropping Parentheses is good:

$$(p \wedge q) \rightarrow p \wedge q$$

But it can lead to ambiguities:

$$((p \vee q) \wedge r) \rightarrow p \wedge q \vee r$$
$$(p \vee (q \wedge r)) \rightarrow p \wedge q \vee r$$

Precedence

Parentheses can be dropped when the structure of an expression can be determined on the basis of precedence.

$$\neg$$

$$\wedge$$

$$\vee$$

$$\Rightarrow \; \Leftarrow \; \Leftrightarrow$$

NB: An operand associates with operator of higher precedence. If surrounded by operators of equal precedence, the operand associates with the operator to the right.

$p \wedge q \vee r$	$p \Rightarrow q \Rightarrow r$	$\neg p \wedge q$
$p \vee q \wedge r$	$p \Rightarrow q \Leftarrow r$	

Propositional Logic Interpretation

A *propositional logic interpretation* is an association between the propositional constants in a propositional language and the truth values T or F.

$$p \xrightarrow{i} \mathrm{T} \qquad p^i = \mathrm{T}$$

$$q \xrightarrow{i} \mathrm{F} \qquad q^i = \mathrm{F}$$

$$r \xrightarrow{i} \mathrm{T} \qquad r^i = \mathrm{T}$$

The notion of interpretation can be extended to all sentences by application of operator semantics.

Operator Semantics

Negation:

ϕ	$\neg\phi$
T	F
F	T

For example, if the interpretation of p is F, then the interpretation of $\neg p$ is T.

For example, if the interpretation of $\neg p$ is T, then the interpretation of $\neg\neg p$ is F.

Operator Semantics (continued)

Conjunction:

ϕ	ψ	$\phi \wedge \psi$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction:

ϕ	ψ	$\phi \vee \psi$
T	T	T
T	F	T
F	T	T
F	F	F

NB: The semantics of disjunction here is often called *inclusive or*, which says that a disjunction is true if and only if *at least* one of its disjuncts is true. This is in contrast with *exclusive or*, according to which a disjunction is true if and only if an odd number of its disjuncts is true. What is the truth table for exclusive or?

Operator Semantics (continued)

Implication:

ϕ	ψ	$\phi \Rightarrow \psi$
T	T	T
T	F	F
F	T	T
F	F	T

Reduction:

ϕ	ψ	$\phi \Leftarrow \psi$
T	T	T
T	F	T
F	T	F
F	F	T

NB: The semantics of implication here is called *material implication*. It has the peculiar characteristic that any implication is true if the antecedent is false, whether or not there is a connection to the consequent. For example, the following is a true sentence.

If George Washington is alive, I am a billionaire.

Operator Semantics (concluded)

Equivalence:

ϕ	ψ	$\phi \Leftrightarrow \psi$
T	T	T
T	F	F
F	T	F
F	F	T

Propositional Logic Interpretations

Logic does not prescribe which interpretation is "correct". In the absence of additional information, one interpretation is as good as another.

Interpretation *i*

$$p^i = \mathrm{T}$$
$$q^i = \mathrm{F}$$
$$r^i = \mathrm{T}$$

Interpretation *j*

$$p^j = \mathrm{F}$$
$$q^j = \mathrm{F}$$
$$r^j = \mathrm{T}$$

Truth Tables

A *truth table* is a table of all possible interpretations for the propositional constants in a language.

p	q	r
1	1	1
1	1	0
1	0	1
1	0	0
0	1	1
0	1	0
0	0	1
0	0	0

One column per constant.

One row per interpretation.

For a language with n constants, there are 2^n interpretations.

Propositional Logic Semantics

The semantics of propositional logic concerns the relationship between interpretations of sentences and interpretations of compound sentences made up of those sentences.

Evaluation:

$$\begin{array}{lcl} p^i & = & \mathrm{T} \\ q^i & = & \mathrm{F} \end{array} \longrightarrow \begin{array}{lcl} (p \vee q)^i & = & \mathrm{T} \\ (\neg q)^i & = & \mathrm{T} \end{array}$$

Disambiguation:

$$\begin{array}{lcl} (p \vee q)^i & = & \mathrm{T} \\ (\neg q)^i & = & \mathrm{T} \end{array} \longrightarrow \begin{array}{lcl} p^i & = & \mathrm{T} \\ q^i & = & \mathrm{F} \end{array}$$

Evaluation

Interpretation *i*:

$$p^i = \text{T}$$
$$q^i = \text{F}$$
$$r^i = \text{T}$$

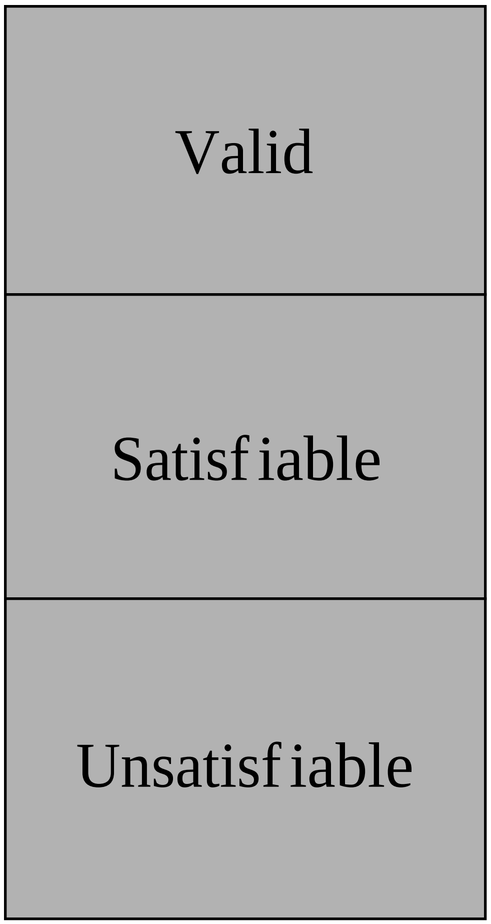
$$(p \vee q) \wedge (\neg q \vee r)$$

Interpretation *j*:

$$p^j = \text{F}$$
$$q^j = \text{F}$$
$$r^j = \text{T}$$

$$(p \vee q) \wedge (\neg q \vee r)$$

Properties of Sentences



A sentence is *valid* if and only if *every* interpretation satisfies it.

A sentence is *satisfiable* if and only if there is *some* interpretation that satisfies it.

A sentence is *unsatisfiable* if and only if *no* interpretation satisfies it.

Example of Validity

p	q	r	$(p \Rightarrow q)$	$(q \Rightarrow r)$	$(p \Rightarrow q) \vee (q \Rightarrow r)$
1	1	1			
1	1	0			
1	0	1			
1	0	0			
0	1	1			
0	1	0			
0	0	1			
0	0	0			

More Validities

Double Negation:

$$p \Leftrightarrow \neg\neg p$$

deMorgan's Laws:

$$\neg(p \wedge q) \Leftrightarrow (\neg p \vee \neg q)$$
$$\neg(p \vee q) \Leftrightarrow (\neg p \wedge \neg q)$$

Implication Introduction:

$$p \Rightarrow (q \Rightarrow p)$$

Implication Distribution

$$(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$$

Disambiguation

By crossing out rows, it is possible to find interpretations implicit in a set of compound sentences.

p	*q*	*r*
1	1	1
1	1	0
1	0	1
1	0	0
0	1	1
0	1	0
0	0	1
0	0	0

Disambiguation

By crossing out rows, it is possible to find interpretations implicit in a set of sentences.

$q \Rightarrow r$

p	q	r	
1	1	1	
1	1	0	×
1	0	1	
1	0	0	
0	1	1	
0	1	0	×
0	0	1	
0	0	0	

Disambiguation

By crossing out rows, it is possible to find interpretations implicit in a set of sentences.

$q \Rightarrow r$

$p \Rightarrow q \wedge r$

p	q	r	
1	1	1	
1	1	0	×
1	0	1	×
1	0	0	×
0	1	1	
0	1	0	×
0	0	1	
0	0	0	

Disambiguation

By crossing out rows, it is possible to find interpretations implicit in a set of sentences.

$q \Rightarrow r$

$p \Rightarrow q \wedge r$

$\neg r$

p	q	r	
1	1	1	$\times$
1	1	0	$\times$
1	0	1	$\times$
1	0	0	$\times$
0	1	1	$\times$
0	1	0	$\times$
0	0	1	$\times$
0	0	0	

The Big Game

Stanford people always tell the truth, and Berkeley people always lie. Unfortunately, by looking at a person, you cannot tell whether he is from Stanford or Berkeley.

You come to a fork in the road and want to get to the football stadium down one fork. However, you do not know which to take. There is a person standing there. What single question can you ask him to help you decide which fork to take?

Basic Idea

left	*su*	*Question*	*Response*
T	T		
T	F		
F	T		
F	F		

The Big Game Solved

Question: *The left road the way to the stadium if and only if you from Stanford. Is that correct?*

$$left \Leftrightarrow su$$